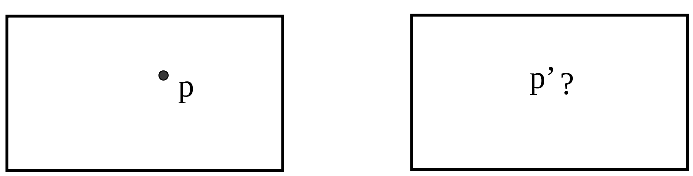
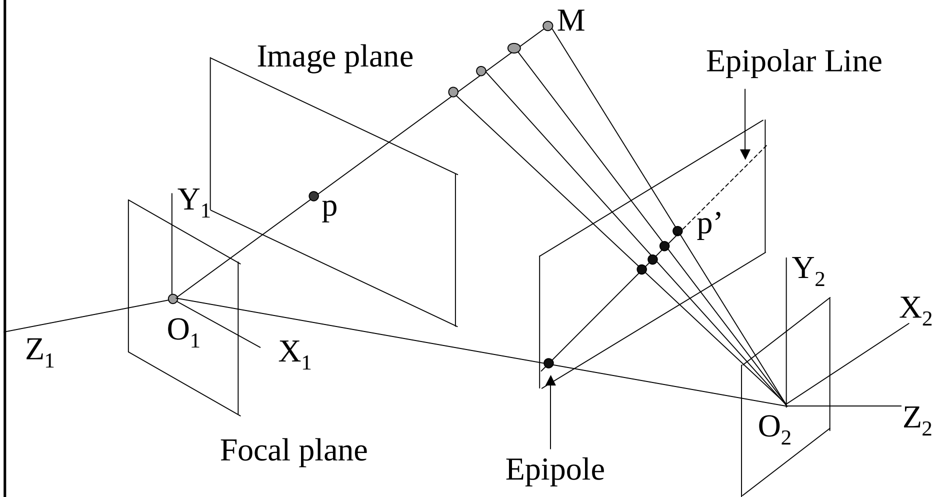
## Multi-View Geometry (Cont.)

## Stereo Constraints (Review)




Given p in left image, where can the corresponding point p’ in right image be?

# Stereo Constraints (Review)




# Epipolar Constraint (Review)

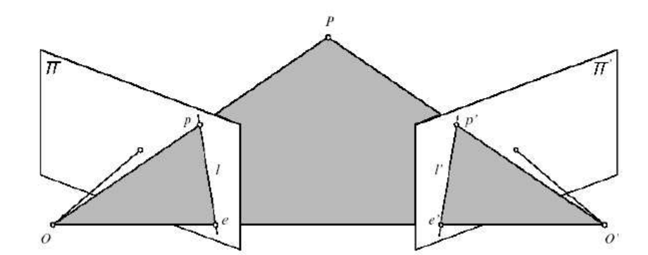



FIGURE 11.1: Epipolar geometry: the point $P$, the optical centers $O$ and $O'$ of the two cameras, and the two images $p$ and $p'$ of $P$ all lie in the same plane.

All epipolar lines contain epipole, the image of other camera center.

## From Geometry to Algebra (Review)

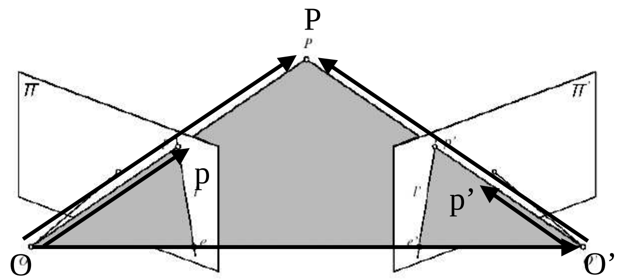
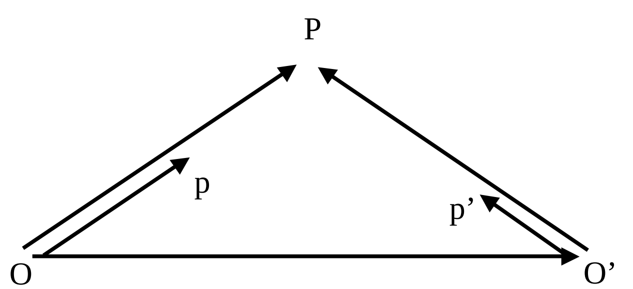


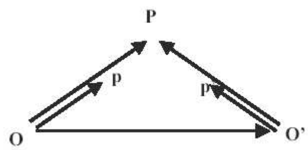

FIGURE 11.1: Epipolar geometry: the point $P$, the optical centers $O$ and $O'$ of the two cameras, and the two images $p$ and $p'$ of $P$ all lie in the same plane.

## From Geometry to Algebra (Review)

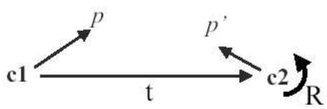



The epipolar constraint: these vectors are coplanar:

$$\overrightarrow{Op} \cdot [\overrightarrow{OO'} \times \overrightarrow{O'p'}] = 0$$




$$\overrightarrow{Op} \cdot [\overrightarrow{OO'} \times \overrightarrow{O'p'}] = 0$$




*p,p' are image coordinates of P in c1 and c2…*

*c2 is related to c1 by rotation R and translation t*

$$\boldsymbol{p} \cdot [\boldsymbol{t} \times (\mathcal{R}\boldsymbol{p'})] = 0$$

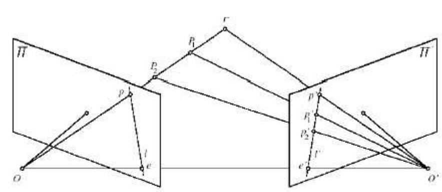
Linear Constraint:
Should be able to express as matrix multiplication.

## The Essential Matrix (Review)

Matrix that relates image of point in one camera to a second camera, given translation and rotation.

$$\varepsilon = [t_x]\Re$$

$$\boldsymbol{p}^T \mathcal{E} \boldsymbol{p}' = 0$$



$$\vec{a} \times \vec{b} = [a_x]\vec{b}$$

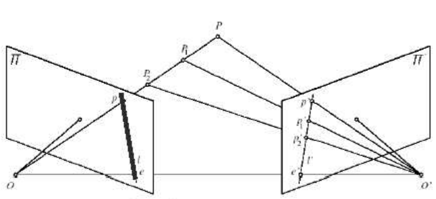
## The Essential Matrix

- Based on the Relative Geometry of the Cameras
- Assumes Cameras are calibrated (i.e., intrinsic parameters are known)
- Relates image of point in one camera to a second camera (points in camera coordinate system).
- Is defined up to scale
- 5 independent parameters

## The Essential Matrix

$\mathcal{E}p'$ is the epipolar line corresponding to p' in the left camera.

$$au + bv + c = 0$$



$\mathcal{E}p'$

$$p = (u, v, 1)^T$$
$$l = (a, b, c)^T$$
$$l \cdot p = 0$$

$$\mathcal{E}p' \cdot p = 0$$
$$\boldsymbol{p}^T \mathcal{E} \boldsymbol{p}' = 0$$

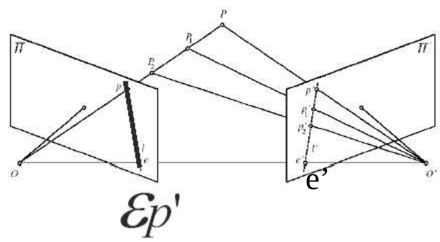
Similarly $\mathcal{E}^T p$ is the epipolar line corresponding to p in the right camera

## The Essential Matrix

$\mathcal{E}e' = [t_\times]\mathrm{R}e' = 0$

Similarly, $\mathcal{E}^T e = R^T [t_\times]^T e = -R^T [t_\times] e = 0$

Essential Matrix is singular with rank 2




## Small Motions and Epipolar Constraint

## Motion Models (Review)

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \approx \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

3D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \approx \left( \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} X'-X \\ Y'-Y \\ Z'-Z \end{bmatrix} \approx \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \approx \begin{bmatrix} 0 & -\omega_Z & \omega_Y \\ \omega_Z & 0 & -\omega_X \\ -\omega_Y & \omega_X & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} V_{T_x} \\ V_{T_Y} \\ V_{T_Z} \end{bmatrix}$$

$\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} =$ Velocity Vector

$\begin{bmatrix} V_{T_X} \\ V_{T_Y} \\ V_{T_Z} \end{bmatrix} =$ Translational Component of Velocity

$\begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} =$ Angular Velocity

## Small Motions

$$t = \delta t v$$

$$R = I + \delta t [\omega_\times]$$

$$p' = p + \delta t \dot{p}$$

$$p^T \mathcal{E} p' = 0$$

$$p^T [v_\times] (I + \delta t [\omega_\times]) (p + \delta t \dot{p}) = 0$$

$$p^T ([v_\times][\omega_\times]) p - (p \times \dot{p}).v = 0$$

$\dot{p} = \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} =$ Velocity Vector

$v = \begin{bmatrix} V_{T_X} \\ V_{T_Y} \\ V_{T_Z} \end{bmatrix} =$ Translational Component of Velocity

$\omega = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} =$ Angular Velocity

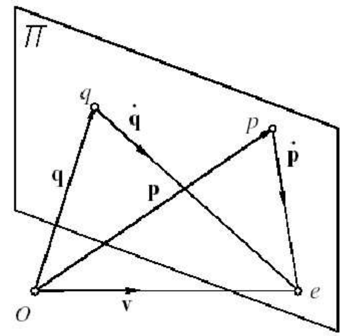
## Translating Camera

$$p^T([v_\times][\omega_\times])p - (p \times \dot{p}).v = 0$$

$$\omega = 0$$

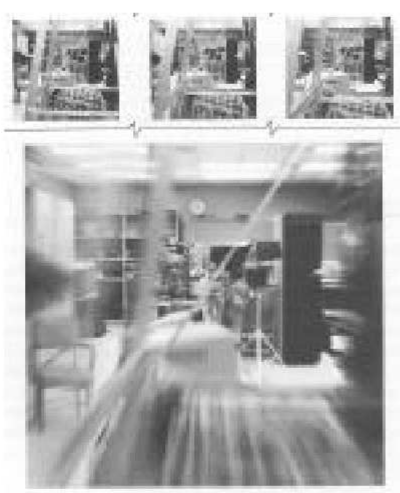
$$(p \times \dot{p}).v = 0$$

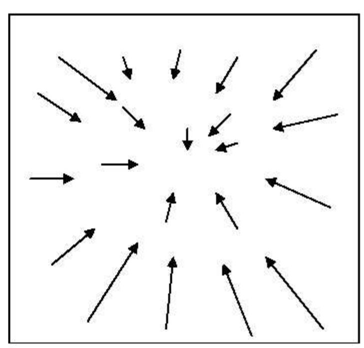
$p$, $\dot{p}$, and $v$ are coplanar




Focus of expansion (FOE): Under pure translation, the motion field at every point in the image points toward the focus of expansion

## FOE for Translating Camera





## FOE from Basic Equations of Motion

$$\dot{p}_x = \frac{V_{T_Z} x - V_{T_X} f}{Z} - \omega_Y f + \omega_Z y + \frac{\omega_X xy}{f} - \frac{\omega_Y x^2}{f}$$

$$\dot{p}_y = \frac{V_{T_Z} y - V_{T_Y} f}{Z} + \omega_X f - \omega_Z x - \frac{\omega_Y xy}{f} + \frac{\omega_X y^2}{f}$$

$$\omega = 0$$

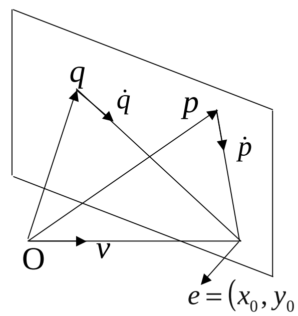
$$\dot{p}_x = \frac{V_{T_Z} x - V_{T_X} f}{Z}$$

$$\dot{p}_y = \frac{V_{T_Z} y - V_{T_Y} f}{Z}$$

$$x_0 = f\frac{V_{T_X}}{V_{T_Z}} \qquad y_0 = f\frac{V_{T_Y}}{V_{T_Z}}$$

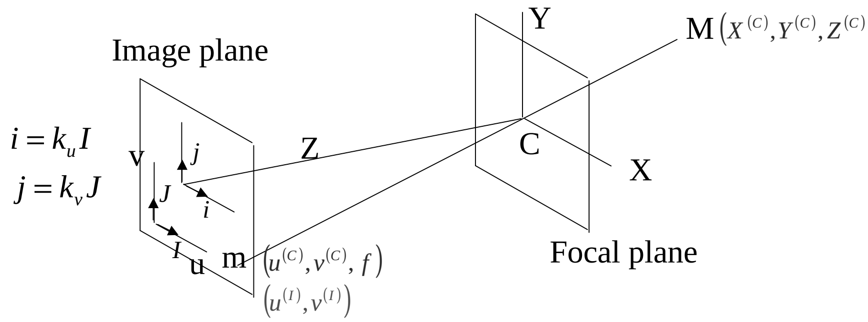
$$\Longrightarrow$$

$$\dot{p}_x = (x - x_0)\frac{V_{T_Z}}{Z} \qquad \dot{p}_y = (y - y_0)\frac{V_{T_Z}}{Z}$$




## What if Camera Calibration is not known

## Review: Intrinsic Camera Parameters




$$\begin{bmatrix} U^{(new)} \\ V^{(new)} \\ S \end{bmatrix} = \underbrace{\begin{bmatrix} -f_u & 0 & u_0 & 0 \\ 0 & -f_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} X^{(C)} \\ Y^{(C)} \\ Z^{(C)} \\ 1 \end{bmatrix} \qquad \begin{aligned} f_u &= fk_u \\ f_v &= fk_v \end{aligned}$$

## Fundamental Matrix

$p^T \mathcal{E} p' = 0$ $p$ and $p'$ are in camera coordinate system

If *u* and *u'* are corresponding image coordinates then we have

$$\begin{aligned} u &= P_1 p \\ u' &= P_2 p' \end{aligned} \Longrightarrow \begin{aligned} p &= P_1^{-1} u \\ p' &= P_2^{-1} u' \end{aligned}$$

$$u^T P_1^{-T} \mathcal{E} P_2^{-1} u' = 0$$

$$\boxed{\Rightarrow u^T F u' = 0} \qquad F = P_1^{-T} \mathcal{E} P_2^{-1}$$

## Fundamental Matrix

$$u^T F u' = 0$$

$$F = P_1^{-T} \mathcal{E} P_2^{-1}$$

Fundamental Matrix is singular with rank 2

In principal F has 7 parameters up to scale and can be estimated from 7 point correspondences

Direct Simpler Method requires 8 correspondences

## Estimating Fundamental Matrix

The 8-point algorithm

$$u^T F u' = 0$$

Each point correspondence can be expressed as a linear equation

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & uv' & u & u'v & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

## The 8-point Algorithm

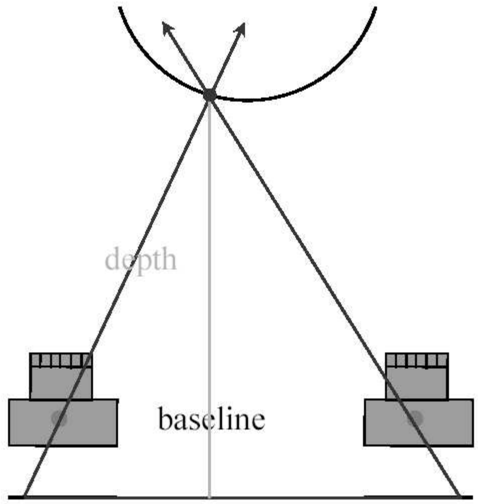
8 corresponding points, 8 equations.

$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' \\ u_3u_3' & u_3v_3' & u_3 & v_3u_3' & v_3v_3' & v_3 & u_3' & v_3' \\ u_4u_4' & u_4v_4' & u_4 & v_4u_4' & v_4v_4' & v_4 & u_4' & v_4' \\ u_5u_5' & u_5v_5' & u_5 & v_5u_5' & v_5v_5' & v_5 & u_5' & v_5' \\ u_6u_6' & u_6v_6' & u_6 & v_6u_6' & v_6v_6' & v_6 & u_6' & v_6' \\ u_7u_7' & u_7v_7' & u_7 & v_7u_7' & v_7v_7' & v_7 & u_7' & v_7' \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

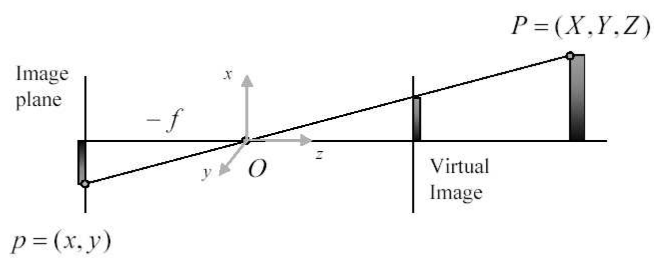
Invert and solve for $\mathcal{F}$.

(Use more points if available; find least-squares solution to minimize $\sum_{i=1}^{n} (\boldsymbol{p}_i^T \mathcal{F} \boldsymbol{p}_i')^2$ )

## Shape from Stereo

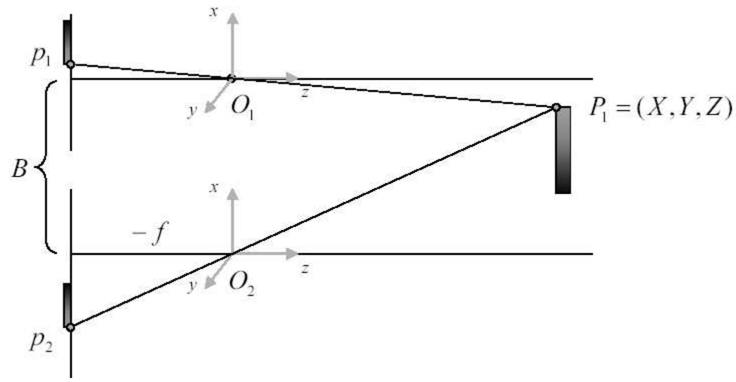



# Pinhole Camera Model



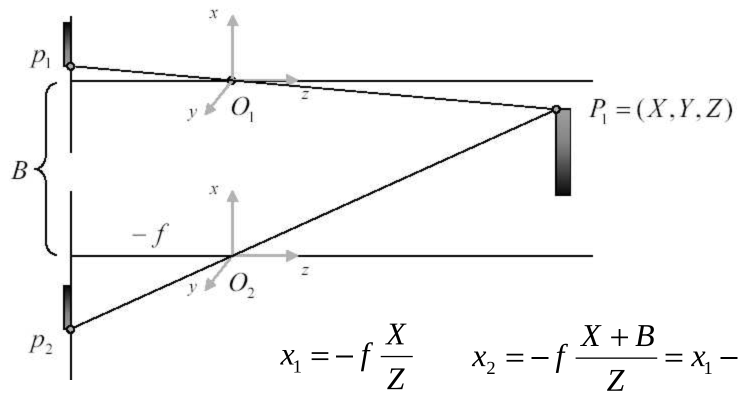

$$x = -f\,\frac{X}{Z}$$

## Basic Stereo Derivations




Derive expression for Z as a function of $x_1$, $x_2$, $f$ and $B$

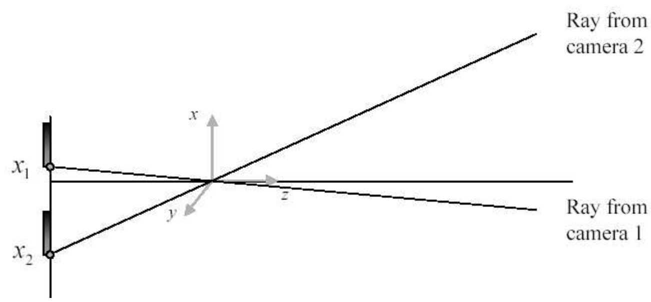
## Basic Stereo Derivations




$$x_1 = -f\,\frac{X}{Z} \qquad x_2 = -f\,\frac{X+B}{Z} = x_1 - f\,\frac{B}{Z}$$

$$Z = \frac{fB}{x_1 - x_2}$$

# Basic Stereo Derivations




Define the disparity: $d = x_1 - x_2$

$$Z = \frac{fB}{d}$$