Preliminaries

Lecture-2

Eigen Vectors and Eigen Values

The eigen vector, x, of a matrix *A* is a special vector, with the following property

$$Ax = lx$$ Where ë is called eigen value

To find eigen values of a matrix A first find the roots of:

$$\det(A - lI) = 0$$

Then solve the following linear system for each eigen value to find corresponding eigen vector

$$(A - lI)x = 0$$

Example

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$

Eigen Values

$$l_1 = 7,\ l_2 = 3,\ l_3 = -1$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix},\ \mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ \mathbf{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen Vectors

Eigen Values

$$\det(A - lI) = 0$$

$$\det\left(\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} - l \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -1-l & 2 & 0 \\ 0 & 3-l & 4 \\ 0 & 0 & 7-l \end{bmatrix}\right) = 0$$

$$(-1-l)((3-l)(7-l)-0) = 0$$

$$(-1-l)(3-l)(7-l) = 0$$

$$l = -1,\quad l = 3,\quad l = 7$$

Eigen Vectors

$l = -1 \qquad (A - lI)x = 0$

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 + 2x_2 + 0 = 0$$
$$0 + 4x_2 + 4x_3 = 0$$
$$0 + 0 + 8x_3 = 0$$
$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$trace(A) = \sum_{i=1}^{n} A_{ii}$$

$$trace(A) = \sum_{i=1}^{n} l_i \text{ where } l_i \text{ are eigen valu es}$$

$$\det(A) = \prod_{i=1}^{n} l_i$$

$\det A = 0$ if and only if A is singular

$\det AB = (\det A)(\det B)$

$$\det A^{-1} = \frac{1}{\det A}$$

$QQ^T = Q^T Q = I$, Q is orthogonal

$Q^{-1} = Q^T$

$\det Q = \det Q^T = \pm 1$

Rotation matrices are Orthogonal (orthonormal) Matrices

$$(R_q^Z)^{-1} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R_q^Z)^{-1} = (R_q^Z)^T$$

$$(R_q^Z)(R_q^Z)^T = I$$

Positive-definite

A symetric n×n matrix is positive definite if

$X^T AX > 0.$

- If all eigenvalues of a symmetric matrix are non-negative, it is said to be Positive semi-definite
- If all eigenvalues of a symmetric matrix are non-negative, it is said to be Positive semi-definite
- If a matrix has both positive and negative eigenvalues, it is said to be indefinite

Matrix Factorization

$A = LU$, LU decomposition, L is a Lower triangular, and U is a upper triangular

$A = CU$, QR decompoistion, C is orthonorma l, U is upper trinagular matrix

Singular Value Decomposition (SVD)

Theorem: Any m by n matrix A, for which $m \geq n$,can be written as

$$A = O_1 \Sigma O_2$$

mxn mxn nxn nxn

Σ is diagonal

O_1 , O_2 are orthogonal

$O_1^T O_1 = O_2^T \ O_2 = I$

Singular Value Decomposition (SVD)

• For some linear systems Ax=b, Gaussian Elimination or LU decomposition does not work, because matrix A is singular, or very close to singular. SVD will not only diagnose for you, but it will solve it.

Singular Value Decomposition (SVD)

If A is square, then O_1, Σ, O_2 are all square.

$$O_1^{-1} = O_1^T$$

$$O_2^{-1} = O_2^T$$

$$\Sigma^{-1} = diag(\frac{1}{w_j})$$

$$A^{-1} = O_2 diag(\frac{1}{w_j}) O_1$$

Singular Value Decomposition (SVD)

The condition number of a matrix is the ratio of the largest of the w_j to the smallest of w_j. A matrix is singular if the condition number is infinite, it is ill-conditioned if the condition number is too large.

Singular Value Decomposition (SVD)

$$Ax = b$$

- If A is singular, some subspace of “x” maps to zero; the dimension of the null space is called “nullity”.
- Subspace of “b” which can be reached by “A” is called range of “A”, the dimension of range is called “rank” of A.

Singular Value Decomposition (SVD)

- If A is non-singular its rank is “n”.
- If A is singular its rank <n.
- Rank+nullity=n

Singular Value Decomposition (SVD)

- SVD constructs orthonormal basses of null space and range.
- Columns of O_1 with non-zero w_j spans range.
- Columns of O_2 with zero w_j spans null space.

Solution of Linear System

- How to solve Ax=b, when A is singular?
- If "b" is in the range of "A" then system has many solutions.
- Replace $\frac{1}{w_j}$ by zero if $w_j = 0$

$$x = O_2[diag(\frac{1}{w_j})]O_1^T b$$

Solution of Linear System

If b is not in the range of A, above eq still gives the solution, which is the best possible solution, it minimizes:

$$r \equiv | Ax - b |$$

Cholesky Factorization

A positive-definite symmetric matrix A can be written:

$$A = LDL^T$$

$$A = LD^{\frac{1}{2}} D^{\frac{1}{2}} L^T = \overline{L}\,\overline{L}^T = R^T R$$

L is unit lower triangular matrix
D is a diagonal matrix with strict
Positive elements
$\overline{L}, \overline{R}$ are general lower triangular
and general upper triangular matrices

Spectral Decomposition of A Symmetric Matrix

$$Au_i = l_i u_i$$

$$A = U \mathrm{A} U^T = \sum_{i=1}^{n} l_i u_i u_i^T$$

Norms

$$\| X \|_1 = \sum_{i=1}^{n} | x_i |, \text{ vector norm}$$

$$\| X \|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = (X^T X)^{\frac{1}{2}}, \text{ vector norm}$$

$$\| \mathbf{A} \| = \max_{\|x\|=1} \| \mathbf{A} X \|, \text{ matrix norm}$$

$$\| \mathbf{A} \|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Norms

Condition number

$$k(A) = \| A \| \| A^{-1} \|$$

The matrix A is well-conidtioned if K(A) is close to one and is not well-conditioned, when K(A) is significantly greater than one.

1-D Functions

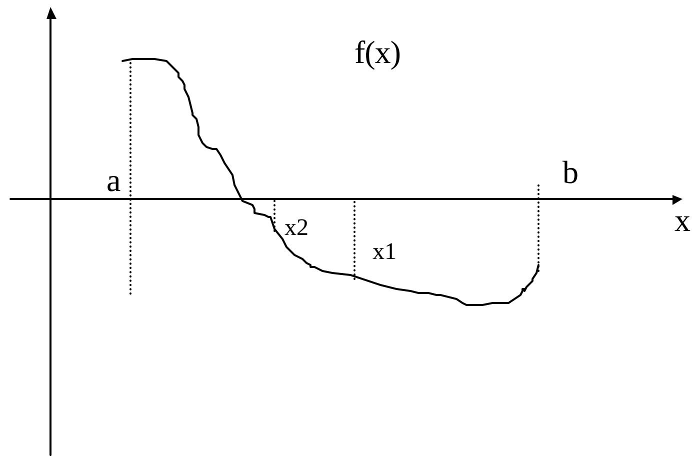
Finding the zero of a function

Bisection Method

- Find a solution to f(x)=0 on the interval [a,b], where f(a)<0 and f(b)>0 have opposite signs.
  - Compute the mid point,m, of [a,b], if f(m)=0, then done
  - else if f(m)>0, then b=m, else a=m

$$| p_n - p | \le \frac{b-a}{2^n}$$

Bisection Method



Newton's Method

Suppose that the function f is twice continously differentiable on the interval $[a,b]$; that is $f \in C^2[a,b]$.

$f'(\bar{x}) \neq 0, |\bar{x} - p|$ is small. Taylor series around $\bar{x}$

$$f(x) = f(\bar{x}) + (x-\bar{x}) f'(\bar{x}) + \frac{(x-\bar{x})^2}{2} f''(x(p)),$$

$x(x)$ lies between x and $\bar{x}$.

$$0 \approx f(\bar{x}) + (p-\bar{x}) f'(\bar{x}) \quad |\bar{x} - p| \text{ is small.}$$

$$p = \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}$$

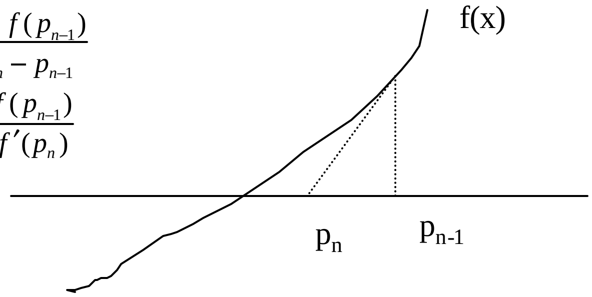
$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Newtons' Methos

$$f'(p_n) = \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$$

$$f'(p_n) = \frac{0 - f(p_{n-1})}{p_n - p_{n-1}}$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)}$$



Zero of the tangent line

Secent Method

If derivative can not be computed
Use finite difference approximation

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Theorem

Let $f \in C^2[a,b]$. If $p \in [a,b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists $d > 0$ such that Netwon's method generates a sequence p_n converging to p for any initial approximation $p_0 \in [p - d, p + d]$.