Lecture-5

Quadratic Functions

Quadratic Functions

$$f(x) = \frac{1}{2}x^T Qx - b^T x$$ Q is symmetric, Hessian of f

$$\nabla f(x) = Qx - b$$

$if\ x^*$ is a unique solution of $Qx = b$, then it is a stationary point of f

If the linear system $Qx=b$ can not be solved, then function does not have a stationary point, it is unbounded

Quadratic Functions

$$f(x) = \frac{1}{2}x^T Qx - b^T x$$ Q is symmetric, Hessian of f

$$\nabla f(x) = Qx - b$$

According to definition, for any vector x and p:

$$f(x + ap) = \frac{1}{2}(x + ap)^T Q(x + ap) - b^T(x + ap)$$

Quadratic Functions

$$f(x + ap) = \frac{1}{2}(x + ap)^T Q(x + ap) - b^T(x + ap)$$

$$f(x + ap) = \frac{1}{2}(x^T Q + ap^T Q)(x + ap) - b^T x - b^T ap$$

$$= \frac{1}{2}(x^T Qx + ap^T Qx + x^T Qap + a^2 p^T Qp) - b^T x - b^T ap$$

$$= \frac{1}{2}x^T Qx - b^T x + \frac{1}{2}(ap^T Qx + x^T Qap + a^2 p^T Qp) - b^T ap$$

$$f(x + ap) = f(x) + ap^T(Qx - b) + \frac{1}{2}a^2 p^T Qp$$

If x^* is stationary point

$$f(x^* + ap) = f(x^*) + ap^T(Qx^* - b) + \frac{1}{2}a^2 p^T Qp$$

$$f(x^* + ap) = f(x^*) + \frac{1}{2}a^2 p^T Qp$$

Quadratic Functions

$$f(x^* + \alpha p) = f(x^*) + \frac{1}{2}\alpha^2 p^T Q p$$

The behavior of f is determine by matrix Q

Let $Qu_j = \lambda_j u_j$

Let p is equal to u_j

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 u_j^T Q u_j$$

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 u_j^T \lambda_j u_j$$

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

Q is symmetric

Quadratic Functions

- The change in f when moving away from x^* along the direction u_j depends on the sign of λ_j
  - If λ_j is positive f will strictly increase as $|\alpha|$ increases
  - If λ_j is negative, f is decreasing as $|\alpha|$ increases.
  - If λ_j is zero, the value of f remains constant when moving along any direction parallel to u_j
  - f reduces to a linear function along any such direction, since quadratic term vanishes.

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

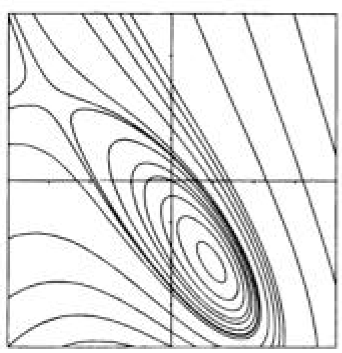
Quadratic Functions

- When all eigenvalues of Q are positive, x^* is the unique global minimum.
  - The contours of f are ellipsoid whose principal axes are in the directions of the eigenvectors of Q, with lengths proportional to square root of corresponding eigenvalues.
- If Q is positive semi-definite, a stationary point (if it exists) is a week local minimum.
- If Q is indefinite and non-singular, x^* is a saddle point, f is unbounded.

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

Iso Contours (Contour Map)

$f(x_1, x_2) = c$



$$f(x_1, x_2) = e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1)$$
$$c = .2, .4, 1, 1.7, 1.8, 2, 3, 4, 5, 6, 20$$

Quadratic Functions

Two positive eigenvalues

$$Q = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -5.5 \\ -3.5 \end{bmatrix}$$

PD

Eigenvalue 6.8541, 0.1459

Eigenvectors
-0.8507 0.5257
-0.5257 -0.8507

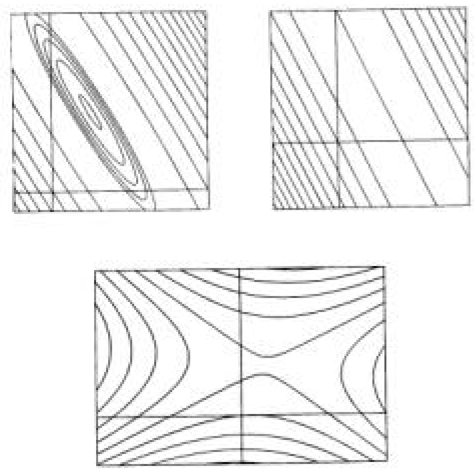


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

One positive eigenvalue,
one zero eigenvalue

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Semi PD

Eigenvalue 5, 0

Eigenvectors
-0.8944 0.4472
-0.4472 -0.8944

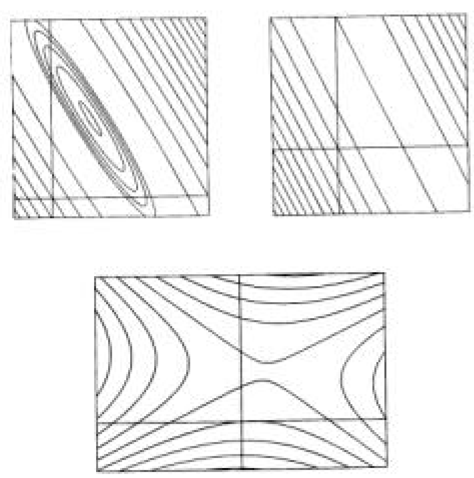


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

One positive eigenvalue,
one zero eigenvalue

$$Q = \begin{bmatrix} 3 & -1 \\ -1 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} -.5 \\ 8.5 \end{bmatrix}$$

Indefinite

Eigenvalue 3.0902, -8.0902

Eigenvectors
-0.9960 -0.0898
0.0898 -0.9960

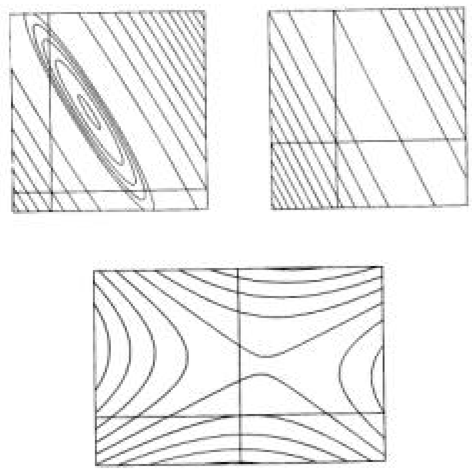


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

How about a function with Q, which is a diagonal matrix?

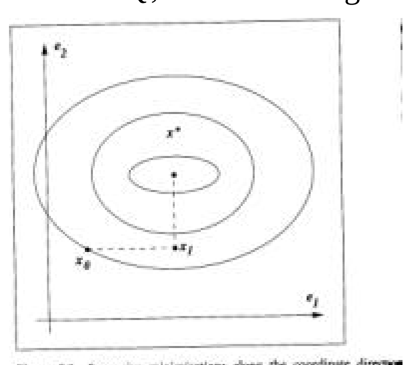
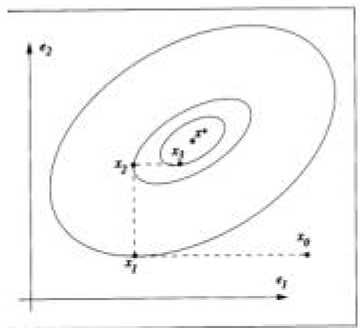


Figure 5.1 Successive minimizations along the coordinate direction minimizer of a quadratic with a diagonal Hessian in n iterations.

Steepest Descent



igure 5.2 Successive minimization along coordinate axes does not find the solution n iterations, for a general convex quadratic.

Quadratic Functions

How about a function with Q, which is a multiple of an identity matrix?

Steepest Descent

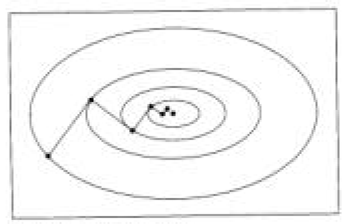


Figure 3.7 Steepest descent steps.

Convergence Rate of Steepest Descent

$$f(x) = \frac{1}{2} x^T Q x - b^T x$$

$$\nabla f(x) = Qx - b$$

x^* is a unique solution of $Qx = b$

Let us compute step length, which minimizes the function:

$$f(x_k - a g_k) = \frac{1}{2}(x_k - a g_k)^T Q (x_k - a g_k) - b^T (x_k - a g_k)$$

Convergence Rate of Steepest Descent

$$\frac{d}{d\alpha} f(x_k - \alpha g_k) = \frac{d}{d\alpha}(\frac{1}{2}(x_k - \alpha g_k)^T Q(x_k - \alpha g_k) - b^T(x_k - \alpha g_k)) = 0$$

$$= -(x_k - \alpha g_k)^T Q g_k + b^T g_k = 0$$

$$-x_k^T Q g_k + \alpha g_k^T Q g_k + b^T g_k = 0$$

$$\alpha g_k^T Q g_k = x_k^T Q g_k - b^T g_k$$

$$\alpha = \frac{x_k^T Q g_k - b^T g_k}{g_k^T Q g_k}$$

$$\alpha = \frac{(x_k^T Q - b^T) g_k}{g_k Q g_k} \qquad \nabla f(x) = Qx - b$$

$$\alpha = \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k}$$

$$x_{k+1} = x_k - \alpha_k \nabla f_k \qquad x_{k+1} = x_k - \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k} \nabla f_k$$

Convergence Rate of Steepest Descent

Define

$$\frac{1}{2} \| x - x^* \|_Q^2 = f(x) - f(x^*)$$

Using: $x_{k+1} = x_k - \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k} \nabla f_k$

It can be shown (homework):

$$\| x_{k+1} - x^* \|_Q^2 = \left\{ 1 - \frac{(\nabla f_k^T \nabla f_k)^2}{(\nabla f_k^T Q \nabla f_k)(\nabla f_k^T Q^{-1} \nabla f_k)} \right\} \| x_k - x^* \|_Q^2$$

OR

$$\| x_{k+1} - x^* \|_Q^2 \leq \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} \right)^2 \| x_k - x^* \|_Q^2$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ are eigenvalues of Q

Convergence Rate of Steepest Descent

$$\| x_{k+1} - x^* \|_Q^2 \le \left(\frac{l_n - l_1}{l_n + l_1} \right)^2 \| x_k - x^* \|_Q^2$$

As the condition number increases the contours of the quadratic become more elongated, the zigzags of line search becomes more pronounced.

Theorem 3.4: Steepest Descent

$$f(x_{k+1}) - f(x^*) \le \left(\frac{l_n - l_1}{l_n + l_1} \right)^2 (f(x_k) - f(x^*))$$

where $0 \le l_1 \le l_2 \le \ldots l_n$ are eigenvalues of Hessian

If the condition number is 800, and $f(x_1)$=1 and $f(x^*)=0$, After 1000 iterations the value of function will be .08.