Lecture-5

Other Optical Flow Methods

Important Issues

- Local vs Global optical flow
- What motion model?
- What function to be minimized?
- What minimization method?

Minimization Methods

- Least Squares fit
- Weighted Least Squares fit
- Newton-Raphson
- Gradient Descent
- Levenberg-Marquadet

Lucas & Kanade (Least Squares)

- Optical flow eq

$$f_x u + f_y v = -f_t$$

- Consider 3 by 3 window

$$f_{x1} u + f_{y1} v = -f_{t1}$$

$$\vdots$$

$$f_{x9} u + f_{y9} v = -f_{t9}$$

$$\begin{bmatrix} f_{x1} & f_{y1} \\ \vdots & \vdots \\ & \\ & \\ f_{x9} & f_{y9} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -f_{t1} \\ \vdots \\ \\ \\ -f_{t9} \end{bmatrix}$$

$$\mathbf{Au} = \mathbf{f_t}$$

Lucas & Kanade

$$\mathbf{Au} = \mathbf{f_t}$$

$$\mathbf{A^T A u} = \mathbf{A^T f_t}$$

$$\mathbf{u} = (\mathbf{A^T A})^{-1} \mathbf{A^T f_t}$$

⇧

$$\min \sum_{i=-2}^{2} \sum_{j=-2}^{2} (f_{xi} u + f_{yi} v + f_{ti})^2$$

Lucas & Kanade

$$\min \sum_{i=-2}^{2} \sum_{j=-2}^{2} (f_{xi} u + f_{yi} v + f_{ti})^2$$

$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{xi} = 0$$

$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{yi} = 0$$

Lucas & Kanade

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum f_{xi}^2 & \sum f_{xi} f_{yi} \\ \sum f_{xi} f_{yi} & \sum f_{yi}^2 \end{bmatrix}^{-1} \begin{bmatrix} -\sum f_{xi} f_{ti} \\ -\sum f_{yi} f_{ti} \end{bmatrix}$$

Lucas & Kanade

$$\min \sum_{i=-2}^{2} \sum_{j=-2}^{2} w_i (f_{xi} u + f_{yi} v + f_{ti})^2$$

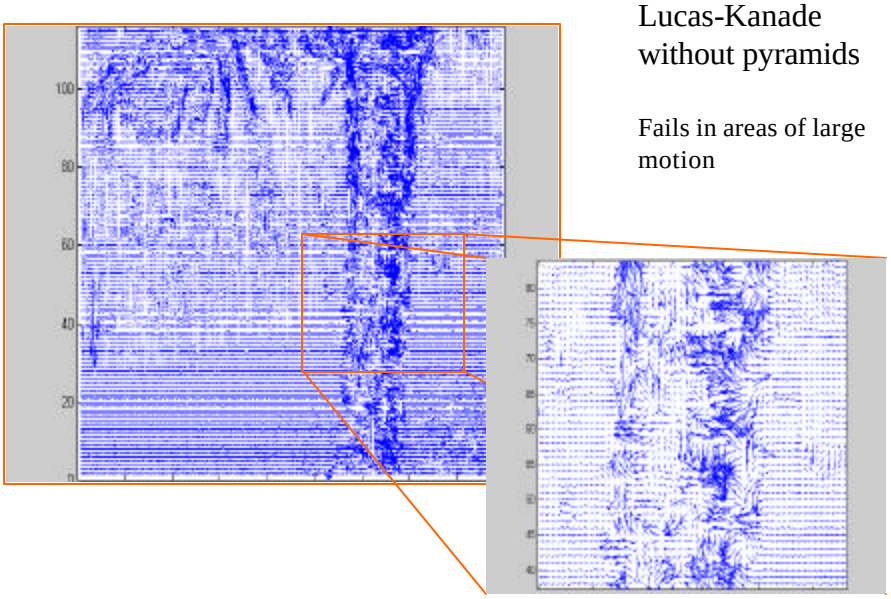
$$\mathbf{WAu} = \mathbf{Wf_t}$$

$$\mathbf{A^T WAu} = \mathbf{A^T Wf_t}$$

$$\mathbf{u} = (\mathbf{A^T WA})^{-1} \mathbf{A^T Wf_t}$$

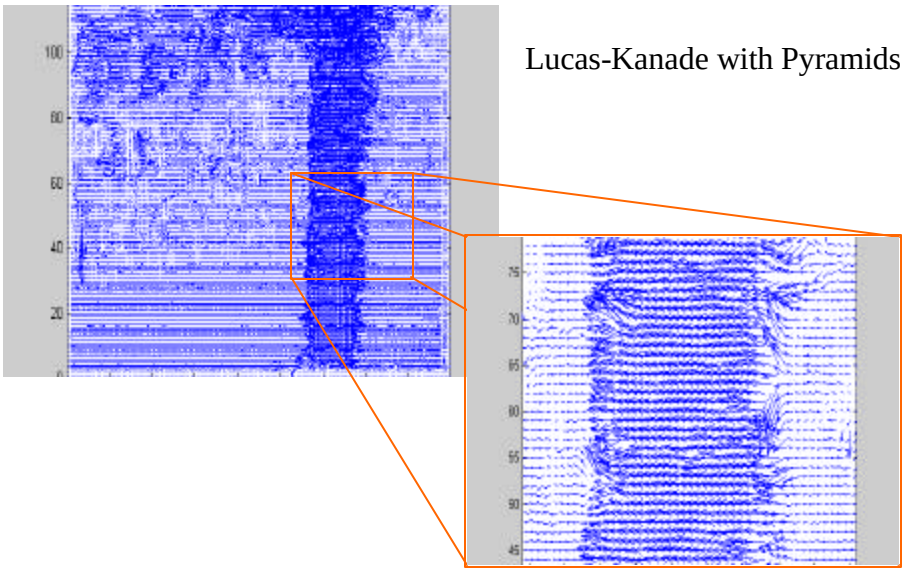
Lucas Kanade with Pyramids

- Compute 'simple' LK at highest level
- At level *i*
  - Take flow u_{i-1}, v_{i-1} from level *i*-1
  - bilinear interpolate it to create u_i^*, v_i^* matrices of twice resolution
  - multiply u_i^*, v_i^* by 2
  - compute f_t from a block displaced by $u_i^*(x,y)$, $v_i^*(x,y)$
  - Apply LK to get $u_i'(x, y)$, $v_i'(x, y)$ (the correction in flow)
  - Add correction to u_{i-1}, v_{i-1} to get u_i , v_i

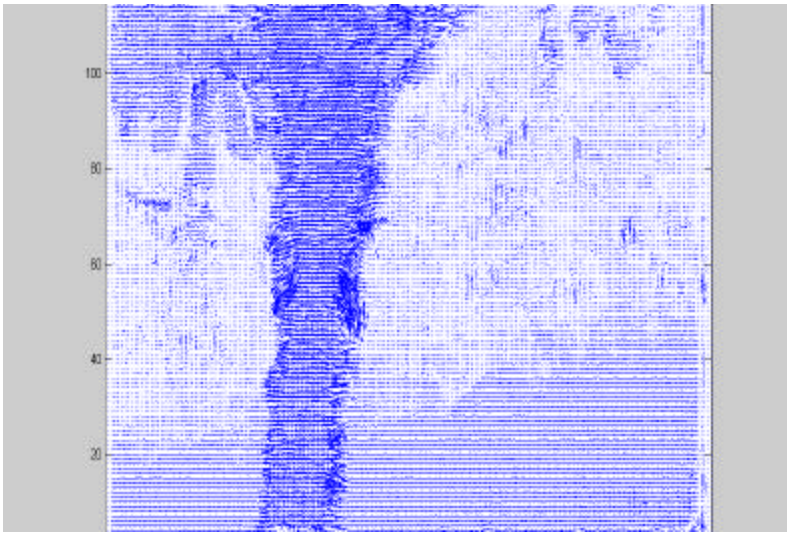


Lucas-Kanade without pyramids

Fails in areas of large motion



Lucas-Kanade with Pyramids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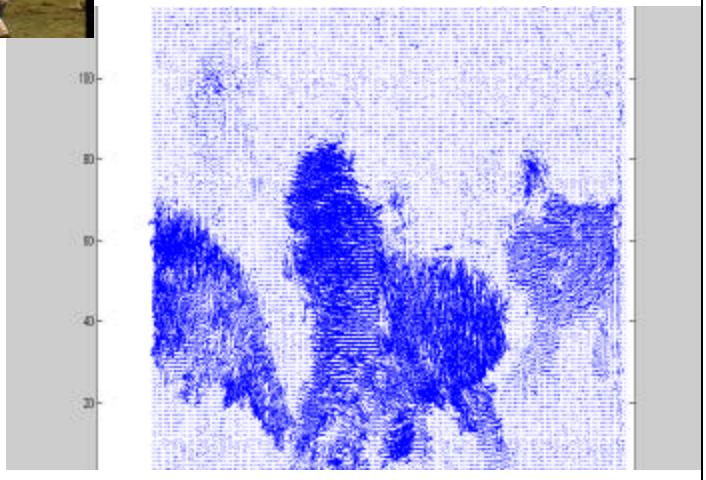
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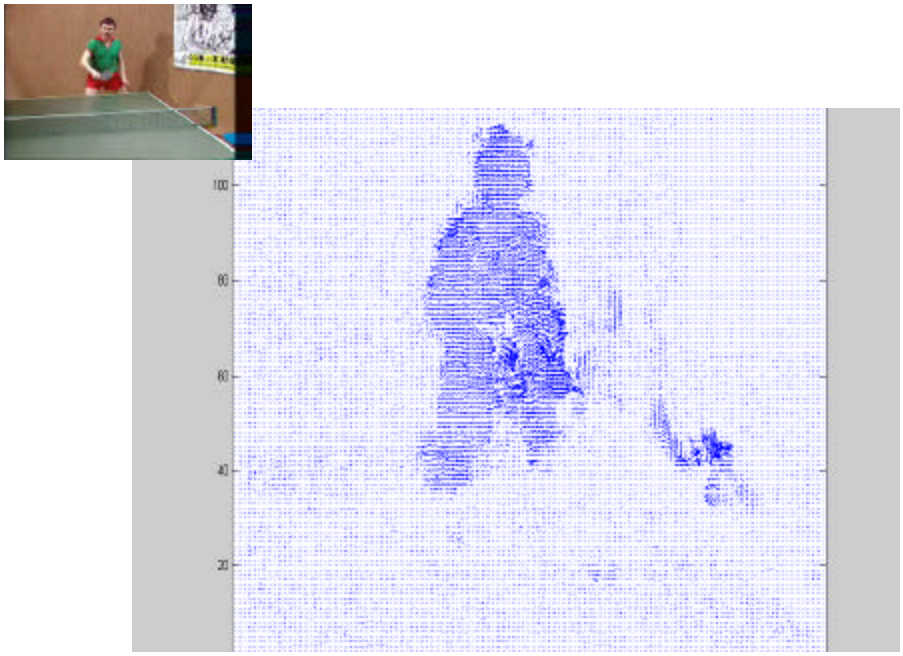
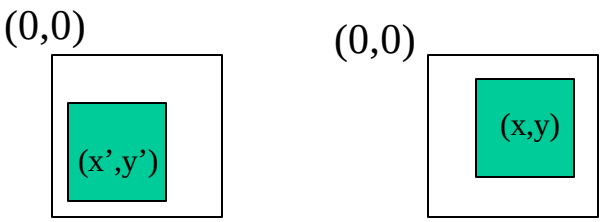


Global Flow

Anandan

Affine

Affine



$$u(x, y) = a_1 x + a_2 y + b_1$$
$$v(x, y) = a_3 x + a_4 y + b_2$$

$$X' = X - U$$

Anandan

$$u(x,y) = a_1 x + a_2 y + b_1$$
$$v(x,y) = a_3 x + a_4 y + b_2$$

•Affine

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ a_3 \\ a_4 \\ b_2 \end{bmatrix}$$

$$\mathbf{u(x) = X(x)a}$$

Anandan

$$\mathbf{u(x) = X(x)a}$$

$$f_{\mathbf{x}} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$E(da) = \sum_x (f_t + f_{\mathbf{x}}^T du)^2$$

Optical flow constraint eq

$$E(da) = \sum_x (f_t + f_{\mathbf{x}}^T \mathbf{X} da)^2$$

$$f_x u + f_y v = -f_t$$

min ⇩ Homework 2: Show this due 9/21

$$\left[\sum \mathbf{X}^{\mathbf{T}} (\mathbf{f_X})(\mathbf{f_x})^T \mathbf{X}\right] d\mathbf{a} = -\sum \mathbf{X}^T \mathbf{f_X} f_t$$

$$Ax = b$$

Basic Components

- Pyramid construction
- Motion estimation
- Image warping
- Coarse-to-fine refinement

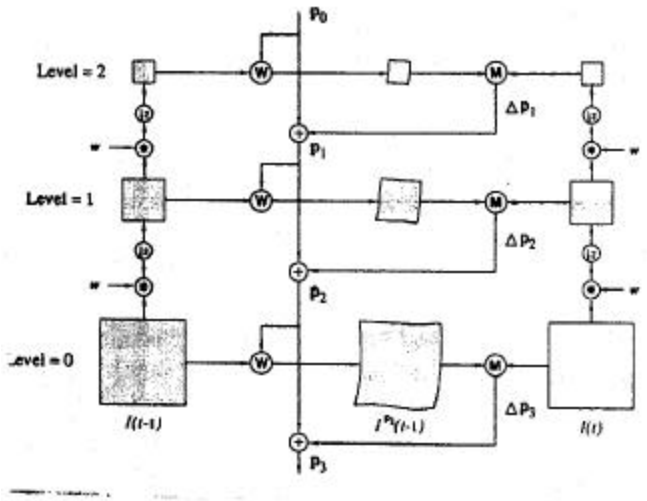
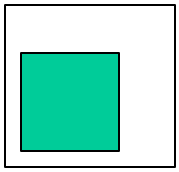
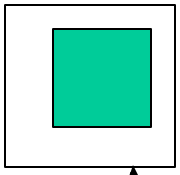


Image Warping

$$X' = X - U$$
$$= X - (AX + b)$$



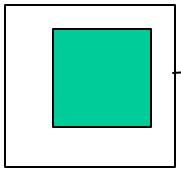
$f(X', t-1)$



$f(X, t)$

warp

$f(X'', t-1)$



$$(I - A)^{-1}(X' + b) = X''$$
$$(A')^{-1}(X' + b) = X''$$

Image Warping

$$X' = X - U = X - (AX + b)$$
$$X' = (I - A)X - b$$
$$X' = A'X - b$$
$$X' + b = A'X$$
$$(A')^{-1}(X' + b) = X$$

Image at time t: X
Image at time t-1: X'

$$(A')^{-1}(X' + b) = X''$$

$$X' \rightarrow X''$$

Image Warping

•How about values in $X'' = (x'', y'')$ are not integer.

•But image is sampled only at integer rows and columns

• Instead of converting X' to X'' and copying $f(X', t-1)$ at $f(X'', t-1)$ we can convert integer values X'' to X' and copy $f(X', t-1)$ at $f(X'', t-1)$

Image Warping

• But how about the values in X' are not integer.

• Perform bilinear interpolation to compute $f(X', t-1)$ at non-integer values.

Image Warping

$$(A')^{-1}(X' + b) = X''$$

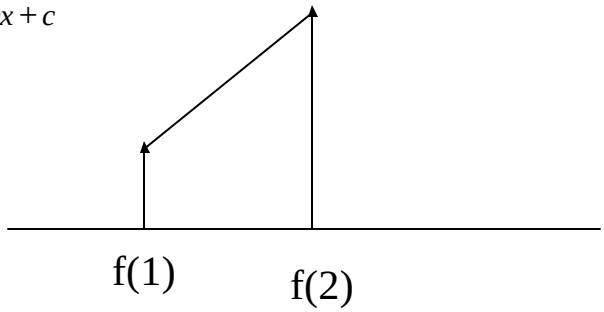
$$(X' + b) = (A')X''$$

$$X' = (A')X'' - b \qquad X'' \rightarrow X'$$

1-D Interpolation

$y = mx + c$

$f(x) = mx + c$



2-D Interpolation

$f(x,y)=a_1+a_2x+a_3y+a_4xy$ Bilinear

X X
O X

Bi-linear Interpolation

Four nearest points of (x,y) are:

$$(\underline{x},\underline{y}),(\overline{x},\underline{y}),(\underline{x},\overline{y}),(\overline{x},\overline{y})$$

$$(3,5),(4,5),(3,6),(4,6)$$

$\underline{x}=\text{int}(x)$ 3

$\underline{y}=\text{int}(y)$ 5

$\overline{x}=\underline{x}+1$ 4

$\overline{y}=\underline{y}+1$ 6

(3.2,5.6)

X (3,6) X (3,6)
X(3,5) X (4,5)

$$f'(x,y) = \overline{\overline{e}_x}\,\overline{\overline{e}_y} f(\underline{x},\underline{y}) + \underline{e}_x\,\overline{\overline{e}_y} f(\bar{x},\underline{y}) + \overline{\overline{e}_x}\,\underline{e}_y f(\underline{x},\bar{y}) + \underline{e}_x\,\underline{e}_y f(\bar{x},\bar{y})$$

$$\overline{\overline{e}_x} = \bar{x} - x$$

$$\overline{\overline{e}_y} = \bar{y} - y$$

$$\underline{e}_x = x - \underline{x}$$

$$\underline{e}_y = y - \underline{y}$$

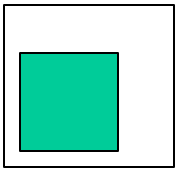
Program-1 Due Oct 3

- (a) Implement Anandan's algorithm using affine transformation. To show the results generate a mosaic (to be discussed next week). Implement all four steps:
  - Pyramid construction
  - Motion estimation
  - Image warping
  - Coarse-to-fine refinement
- (b) Modify program in (a) to use psuedo perspective transformation instead of affine.

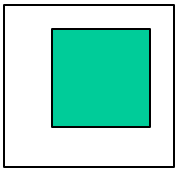
Szeliski

Projective

Projective



$f(X',t-1)$



$f(X,t)$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_1}{c_1 x + c_2 y + 1}$$

Szeliski (Levenberg-Marquadet)

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$ Projective

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

min

Szeliski (Levenberg-Marquadet)

Motion Vector:

$$\mathbf{m} = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad b_1 \quad b_2 \quad c_1 \quad c_2]^T$$

Szeliski (Levenberg-Marquadet)

Hessian $a_{kl} = \sum \frac{\partial e}{\partial m_k} \frac{\partial e}{\partial m_l}$ $b_k = -\sum e \frac{\partial e}{\partial m_k}$ **gradient**

$$\Delta m = (A + lI)^{-1} b$$

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'} \frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'} \frac{\partial y'}{\partial a_1}$$

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

Hessian

$$A=\begin{bmatrix}
\sum_x \frac{\partial^2 e}{\partial a_1 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial a_1 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial a_2 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial a_2 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial a_3 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial a_3 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial a_4 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial a_4 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial b_1 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial b_2 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial b_1} & \frac{\partial^2 e}{\partial b_2 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial b_2 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial c_1 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial c_1 \partial c_2} \\
\sum_x \frac{\partial^2 e}{\partial c_2 \partial a_1} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial a_2} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial a_3} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial a_4} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial b_1} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial b_2} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial c_1} & \sum_x \frac{\partial^2 e}{\partial c_2 \partial c_2}
\end{bmatrix}$$

$$b=\begin{bmatrix}
-\sum_x e\frac{\partial e}{\partial a_1} \\
-\sum_x e\frac{\partial e}{\partial a_2} \\
-\sum_x e\frac{\partial e}{\partial a_3} \\
-\sum_x e\frac{\partial e}{\partial a_4} \\
-\sum_x e\frac{\partial e}{\partial b_1} \\
-\sum_x e\frac{\partial e}{\partial b_2} \\
-\sum_x e\frac{\partial e}{\partial c_1} \\
-\sum_x e\frac{\partial e}{\partial c_2}
\end{bmatrix}$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}, \quad y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_1} = \frac{x}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_1} = 0$$

$$\frac{\partial x'}{\partial a_2} = \frac{y}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_2} = 0$$

$$\frac{\partial x'}{\partial a_3} = 0 \qquad \frac{\partial y'}{\partial a_3} = \frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_4} = 0 \qquad \frac{\partial y'}{\partial a_4} = \frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial b_1} = \frac{1}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial b_1} = 0$$

$$\frac{\partial x'}{\partial b_2} = 0 \qquad \frac{\partial y'}{\partial b_2} = \frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial c_1} = \frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_1} = \frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial x'}{\partial c_2} = \frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_2} = \frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_1} = f_{x'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_2} = f_{x'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_3} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_3} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_3} = f_{y'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_4} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_4} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_4} = f_{y'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_1} = f_{x'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_2} = f_{y'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial c_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_1} = f_{x'}\frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial e}{\partial c_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_2} = f_{x'}\frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\mathbf{b} = \begin{bmatrix} -\sum ef_{x'} \frac{x}{c_1x + c_2y + 1} \\ -\sum ef_{x'} \frac{y}{c_1x + c_2y + 1} \\ -\sum ef_{y'} \frac{x}{c_1x + c_2y + 1} \\ -\sum ef_{y'} \frac{y}{c_1x + c_2y + 1} \\ -\sum ef_{x'} \frac{1}{c_1x + c_2y + 1} \\ -\sum ef_{y'} \frac{1}{c_1x + c_2y + 1} \\ \sum ex \left[\frac{f_{x'}(a_1x + a_2y + b_1) + f_{y'}(a_3x + a_4y + b_2)}{(c_1x + c_2y + 1)^2} \right] \\ \sum ey \left[\frac{f_{x'}(a_1x + a_2y + b_1) + f_{y'}(a_3x + a_4y + b_2)}{(c_1x + c_2y + 1)^2} \right] \end{bmatrix}$$

Szeliski (Levenberg-Marquadet)

- Start with some initial value of m, and ë=.001
  - For each pixel I at (x_i, y_i)
    - Compute (x', y') using projective transform.
    - Compute $e = f(x', y') - f(x, y)$
    - Compute $\frac{\partial e}{\partial m_k} = \frac{\partial e}{\partial x'} \frac{\partial x'}{\partial m_k} + \frac{\partial e}{\partial y'} \frac{\partial y'}{\partial m_k}$

Szeliski (Levenberg-Marquadet)

-Compute A **and** b

-Solve system

$$(A - lI)\Delta m = b$$

-Update

$$m^{t+1} = m^t + \Delta m$$

Szeliski (Levenberg-Marquadet)

• check if error has decreased, if not increase l by a factor of 10 and compute a new Δm

• If error has decreased, decrease by a factor of 10 and compute a new Δm

• Continue iteration until error is below threshold.