Fun With Summations

Computer Science Department
University of Central Florida

COP 3502 – Computer Science I

Summations

- Is this a Math class?
- Why do we study summations?
  - In order to effectively approximate algorithms
  - We NEED mathematical tools
    - It is not always as simply as doing a 4 second examination of a for loop and deciphering the Big-O time
  - So for iterative algorithms
    - We use summations as the tool (discussed today)
  - For recursive algorithms, this doesn't work
    - We need yet another tool
    - Recurrence relations (coming after the exam)

Summations

- Definition:
  - In very basic terms, a summation is the addition of a set of numbers.
- Example:
  - Let's say we want to sum the integers from 1 to 5

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

  - Here, the "i" is just a variable.
  - Let's look at this notation in more detail…

Summations

- A summation:

So what is f(j)?

upper limit

$$\sum_{j=m}^{n} f(j) \quad \text{or} \quad \sum\nolimits_{j=m}^{n} f(j)$$

lower limit

index of summation

- Here, f(j) is simply a function in terms of j
  - Just like f(x) = 2x+1 is a function in terms of x.
  - f(j) is simply some function in terms of j.

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += f(j);
```

Summations

- Example con’d:
  - Since “i” is just a variable, we can use any variable name…

$$\sum_{Jason=1}^{5} Jason = 1 + 2 + 3 + 4 + 5 = 15$$

  - We also recognize that a summation is merely summing (adding) the values of some given function
    - This far, we’ve only looked at this most simply function:
      - f(i) = i
      - And then we summed up those i terms.

Summations

- Example 2:
  - Now, let us choose our function to be i^2, and let us again sum this from 1 to 5.

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

    - So, the deal here is we are summing whatever the function is, and we are summing this from the lower limit all the way to the upper limit, adding all the values together
  - What if we let the function be 2i+1

$$\sum_{i=1}^{5} 2i + 1 = 3 + 5 + 7 + 9 + 11 = 35$$

Summations

- More summations:
  - Now let us write this purely in "function" form so we all see what is going on.

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5)$$

  - Again, on the previous example, we let f(i) = 2i+1
  - Thus, we had…

$$\sum_{i=1}^{5} 2i+1 = (2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)$$

$$= 3+5+7+9+11 = 35$$

Summations

- Example Problems:
  - We now give several example problems
  - For each problem, we give you the summation RULES that you need for the specific problem
  - These are RULES you will need to know for exams
    - Learn ‘em, memorize ‘em, or include them on your 1 page sheet
  - You then use the rules to solve the problem

Summations

- Problem 1. Evaluate: $\sum_{i=0}^{3}(5+\sqrt{4^i})$

Use: $\sum_{i=1}^{n} 1 = n$ $\sum_{i=m}^{n} f(i) + g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$

$$\sum_{i=0}^{3}(5+\sqrt{4^i}) = (5+\sqrt{4^0}) + (5+\sqrt{4^1}) + (5+\sqrt{4^2}) + (5+\sqrt{4^3})$$

$$= (5+\sqrt{1}) + (5+\sqrt{4}) + (5+\sqrt{16}) + (5+\sqrt{64})$$

= (5+1) + (5+2) + (5+4) + (5+8)
= 6 + 7 + 9 + 13

= 35

Summations

- Problem 2. Evaluate: $\sum_{i=1}^{100}(4+3i)$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{100}(4+3i) = \sum_{i=1}^{100} 4 + \sum_{i=1}^{100} 3i$$

$$= \sum_{i=1}^{100} 4 + 3\left(\sum_{i=1}^{100} i\right)$$

$$= 4(100) + 3\left\{\frac{100(100+1)}{2}\right\}$$

= 400 + 15,150

= 15,550

Summations

- Problem 3. Evaluate: $\sum_{i=1}^{200}(i-3)^2$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{200}(i-3)^2 = \sum_{i=1}^{200}(i^2-6i+9)$$

$$\sum_{i=1}^{n} i\wedge 2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \quad = \sum_{i=1}^{200} i^2 - 6\left(\sum_{i=1}^{200} i\right) + \sum_{i=1}^{200} 9$$

$$= \frac{200(200+1)(400+1)}{6} - 6\left\{\frac{200(200+1)}{2}\right\} + 9(200)$$

= 2,686,700 - 120,600 + 1800

= 2,567,900.

Summations

- Problem 4. Evaluate: $\sum_{i=15}^{150}(4i+1)$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=15}^{150}(4i+1) = \sum_{i=15}^{150} 4i + \sum_{i=15}^{150} 1$$

$$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

$$= 4\left(\sum_{i=15}^{150} i\right) + \sum_{i=15}^{150} 1 \qquad = 4\left(\sum_{i=1}^{150} i - \sum_{i=1}^{14} i\right) + \left(\sum_{i=1}^{150} 1 - \sum_{i=1}^{14} 1\right)$$

$$= 4\left(\frac{150(150+1)}{2} - \frac{14(14+1)}{2}\right) + ((1)(150) - (1)(14))$$

= 4(11,325 - 105) + (136)

= 45,016.

Summations

- Problem 5. Evaluate: $\sum_{i=10}^{80}(i^3+i^2)$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

$$\sum_{i=10}^{80}(i^3+i^2) = \sum_{i=10}^{80} i^3 + \sum_{i=10}^{80} i^2$$

$$= \left(\sum_{i=1}^{80} i^3 - \sum_{i=1}^{9} i^3\right) + \left(\sum_{i=1}^{80} i^2 - \sum_{i=1}^{9} i^2\right)$$

$$= \left(\frac{80^2(80+1)^2}{4} - \frac{9^2(9+1)^2}{4}\right) + \left(\frac{80(80+1)(160+1)}{6} - \frac{9(9+1)(18+1)}{6}\right)$$

= 10,497,600 - 2025 + 173,880 - 285

= 10,669,170.

Summations

- Problem 6. Evaluate: $$\sum_{i=2}^{5} 2^i$$

Use: $$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \qquad \sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

$$\sum_{i=2}^{5} 2^i = \sum_{i=1}^{5} 2^i - \sum_{i=1}^{1} 2^i$$

$$= (\sum_{i=0}^{5} 2^i) - 1 - (\sum_{i=0}^{1} 2^i) - 1$$

Now simply apply the rule!

Summations

- Problem 7. Evaluate: $$\sum_{i=1}^{n} \sum_{j=1}^{m} c$$

  Use: $$\sum_{i=1}^{n} k = k * n$$

  - Start with the right-most summation
    - Apply the rule above

      $$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

  - Now, think about something:
    - How did we know that c was a constant
      - Well, at least in part, cuz I told you!
      - In addition to this fact, you know it is a constant because **variables** are always the **SAME letter** as the **index** of the summation
      - j was the index of the inner summation
      - Thus, any letter would be a constant if it was other than the letter j

Summations

- Problem 7. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{m} c$

  Use: $\sum_{i=1}^{n} k = k * n$

  - Start with the right-most summation
    - Apply the rule above

      $$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

  - Now, look at c*m
    - We know c is a constant, but what about m?
    - Well, the index of the summation is i
    - Any other letter, INSIDE the summation simply acts as a constant
    - So c*m is also a constant

Summations

- Problem 7. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{m} c$

  Use: $\sum_{i=1}^{n} k = k * n$

  - Start with the right-most summation
    - Apply the rule above

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

  - So we apply the same rule again:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m = (c * m) * n$$

Summations

- Problem 8. Evaluate: $$\sum_{i=1}^{n}\sum_{j=1}^{i} c$$

Use: $\sum_{i=1}^{n} k = k * n$

Use: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  - Start with the right-most summation
    - Apply the rule above

$$\sum_{i=1}^{n}\sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i$$

  - Now, refer back to the previous example
  - Ask yourself, is c*i a constant?
  - Answer: you must look at the INDEX of the summation
    - i is the index, AND i is in the "body" of the summation
    - Therefore, i is most certainly a variable and should be treated as such!

Summations

- Problem 8. Evaluate: $\sum_{i=1}^{n}\sum_{j=1}^{i} c$

Use: $\sum_{i=1}^{n} k = k * n$

Use: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  - Start with the right-most summation
    - Apply the rule above

$$\sum_{i=1}^{n}\sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i$$

  - Now, we pull out the constant, and apply the next rule:

$$\sum_{i=1}^{n}\sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i = c\sum_{i=1}^{n} i = c\left[\frac{n(n+1)}{2}\right]$$

Evaluating Summations

- Examples:
  - (work out on your own and check answers)

$$\sum_{k=1}^{5}(k+1)$$

- 2 + 3 + 4 + 5 + 6 = 20

$$\sum_{j=0}^{4}(-2)^{j}$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10}3$$

- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$$\sum_{j=0}^{8}\left(2^{j+1}-2^{j}\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \dots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Summation Problems

- Summation Worksheet:
  - http://www.math.niu.edu/~richard/Math229/sum_prac.pdf
- Solutions:
  - http://www.math.niu.edu/~richard/Math229/sums_solns.pdf
- Not all these problems are applicable
  - Such as the Trig ones, etc.
  - Just practice those that are like the ones we discussed

- Another website with summations and solutions:
  - http://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/summationdirectory/Summation.html

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