Recurrence Relations

$$T(n) = T(n-1) + f(n), \quad T(0) = 0$$

$$= T(n-2) + f(n-1) + f(n)$$

$$= T(n-3) + f(n-2) + f(n-1) + f(n)$$

$$\vdots$$

$$= T(0) + f(1) + f(2) + \cdots + f(n)$$

$$= \sum_{i=1}^{n} f(i)$$

$$T(n) = 2T\left(\tfrac{n}{2}\right) + 1, \quad T(1) = 1$$

$$= 2\left[2T\left(\tfrac{n}{4}\right)+1\right] + 1$$

$$= 4T\left(\tfrac{n}{4}\right) + 2 + 1$$

$$= 4T\left(\tfrac{n}{4}\right) + 3$$

$$= 4\left(2T\left(\tfrac{n}{8}\right) + 1\right) + 3$$

$$= 8T\left(\tfrac{n}{8}\right) + 4 + 3$$

$$= 8T\left(\tfrac{n}{8}\right) + 7$$

k steps $\quad = 2^k T\left(\tfrac{n}{2^k}\right) + (2^k - 1)$

$$= nT(1) + (n-1)$$

$$= n + n - 1 = 2n - 1 = \boxed{O(n)}$$

$$T\left(\tfrac{n}{2}\right) = 2T\left(\tfrac{n}{4}\right) + 1$$

$$T\left(\tfrac{n}{4}\right) = 2T\left(\tfrac{n}{8}\right) + 1$$

Let $\frac{n}{2^k} = 1$

$$\boxed{n = 2^k}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2, \quad T(1) = 1$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

$$= 2\left[2T\left(\frac{n}{4}\right) + \frac{n^2}{4}\right] + n^2$$

$$= 4T\left(\frac{n}{4}\right) + \frac{n^2}{2} + n^2$$

$$= 4T\left(\frac{n}{4}\right) + \left(1 + \frac{1}{2}\right)n^2$$

$$= 4\left[2T\left(\frac{n}{8}\right) + \frac{n^2}{16}\right] + \left(1 + \frac{1}{2}\right)n^2$$

$$= 8T\left(\frac{n}{8}\right) + \frac{n^2}{4} + \left(1 + \frac{1}{2}\right)n^2$$

$$= 8T\left(\frac{n}{8}\right) + \left(1 + \frac{1}{2} + \frac{1}{4}\right)n^2$$

k steps $= 2^k T\left(\frac{n}{2^k}\right) + \underline{\left(2 - \left(\frac{1}{2}\right)^{k-1}\right)} n^2$

$$\frac{n}{2^k} = 1$$

$$n = 2^k$$

$$\leq nT(1) + 2n^2$$

$$= n + 2n^2 = \boxed{O(n^2)}$$

$$T(n) = \underline{4T\left(\frac{n}{2}\right) + n}, \quad T(1) = 1$$

$$T\left(\frac{n}{2}\right) = 4T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$T\left(\frac{n}{4}\right) = 4T\left(\frac{n}{8}\right) + \frac{n}{4}$$

$$= 4\left[4T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$$

$$= 16T\left(\frac{n}{4}\right) + 2n + n$$

$$= \underline{16T\left(\frac{n}{4}\right) + 3n}$$

$$= 16\left[4T\left(\frac{n}{8}\right) + \frac{n}{4}\right] + 3n$$

$$= 64T\left(\frac{n}{8}\right) + 4n + 3n$$

$$= \underline{64T\left(\frac{n}{8}\right) + 7n}$$

$$\frac{n}{2^k} = 1$$

k steps $= 4^k T\left(\frac{n}{2^k}\right) + (2^k - 1)n$

$$n = 2^k$$

$$n^2 = 2^k \cdot 2^k$$

$$n^2 = 4^k$$

$$= n^2 T(1) + (n-1)n$$

$$= n^2 + n^2 - n$$

$$= 2n^2 - n = \boxed{O(n^2)}$$

$$T(n) = \underline{2T(\tfrac{n}{4}) + 1}, \quad T(1) = 1$$

$$= 2\left[2T(\tfrac{n}{16}) + 1\right] + 1$$

$$= 4T(\tfrac{n}{16}) + 2 + 1$$

$$= \underline{4T(\tfrac{n}{16}) + 3}$$

$$= 4\left[2T(\tfrac{n}{64}) + 1\right] + 3$$

$$= 8T(\tfrac{n}{64}) + 4 + 3$$

$$= \underline{8T(\tfrac{n}{64}) + 7}$$

k steps $= 2^k T(\tfrac{n}{4^k}) + (2^k - 1)$

$$= \sqrt{n}\,T(1) + \sqrt{n} - 1$$

$$= 2\sqrt{n} - 1 = \boxed{O(\sqrt{n})}$$

$$T(\tfrac{n}{4}) = 2T(\tfrac{n}{16}) + 1$$

$$T(\tfrac{n}{16}) = 2T(\tfrac{n}{64}) + 1$$

Let $\frac{n}{4^k} = 1$

$$n = 4^k$$

$$n = 2^{2k}$$

$$n = (2^k)^2$$

$$\sqrt{n} = 2^k$$

$$T(n) = \underline{3T(\tfrac{n}{2}) + n^2}, \quad T(1) = 1$$

$$= 3\left[3T(\tfrac{n}{4}) + \tfrac{n^2}{4}\right] + n^2$$

$$= \underline{9T(\tfrac{n}{4}) + (1 + \tfrac{3}{4})n^2}$$

$$= 9\left[3T(\tfrac{n}{8}) + \tfrac{n^2}{16}\right] + (1 + \tfrac{3}{4})n^2$$

$$= 27T(\tfrac{n}{8}) + \tfrac{9}{16}n^2 + (1 + \tfrac{3}{4})n^2$$

$$= \underline{27T(\tfrac{n}{8}) + (1 + \tfrac{3}{4} + \tfrac{9}{16})n^2}$$

k steps $= 3^k T(\tfrac{n}{2^k}) + \left(\sum_{i=0}^{k-1} (\tfrac{3}{4})^i\right) n^2$

$$\leq 3^{\log_2 n} T(1) + 4n^2$$

$$= n^{\log_2 3} + 4n^2 = \boxed{O(n^2)}$$

Hints:

$\sum_{i=0}^{\infty} (\tfrac{3}{4})^i = 4$ ✓

$3^{\log_2 n} = n^{\log_2 3}$ ✓

$\log_2 3 < 2$ ✓

$$T(\tfrac{n}{2}) = 3T(\tfrac{n}{4}) + (\tfrac{n}{2})^2$$

$$T(\tfrac{n}{4}) = 3T(\tfrac{n}{8}) + (\tfrac{n}{4})^2$$

Let $\frac{n}{2^k} = 1$

$$n = 2^k$$

$$k = \log_2 n$$

Master Thm

$$T(n) = A\,T\left(\frac{n}{B}\right) + O(n^k)$$

Case1: if $B^k > A$ $(\log_B A < k)$, then $O(n^k)$

Case2: if $B^k = A$ $(\log_B A = k)$, then $O(n^k \lg n)$

Case3: if $B^k < A$ $(\log_B A > k)$, then $O(n^{\log_B A})$

(1) $T(n) = 2T(\frac{n}{2}) + 1$, $A=2, B=2, k=0$
$2^0 = 1 < 2$ Case 3 $O(n^1)$ ✓

(2) $T(n) = 2T(\frac{n}{2}) + n^2$, $A=2, B=2, k=2$
$2^2 = 4 > 2$ Case1 $O(n^2)$ ✓

(3) $T(n) = 4T(\frac{n}{2}) + n$, $A=4, B=2, k=1$
$2^1 < 4$ Case 3 $O(n^2)$ ✓

(4) $T(n) = 2T(\frac{n}{4}) + 1$, $A=2, B=4, k=0$
$4^0 = 1 < 2$ Case 3 $O(n^{\frac{1}{2}})$ ✓

(5) $T(n) = 3T(\frac{n}{2}) + n^2$, $A=3, B=2, k=2$
$2^2 = 4 > 3$ Case1 $O(n^2)$ ✓

*(6) $T(n) = 4T(\frac{n}{2}) + n^2$, $A=4, B=2, k=2$
$2^2 = 4$ ✓ Case2 $O(n^2 \lg n)$

→ WORK OUT VIA ITERATION