COP 3502 3/2/2022

Recurrence Relations Day #2

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2 \quad , \quad T(1) = 1$$

$$= 3\left[3T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$= 9T\left(\frac{n}{4}\right) + \left[\frac{3}{4}n^2 + n^2\right]$$

$$= 9\left[3T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + \left[\frac{3}{4}n^2 + n^2\right]$$

$$= 27T\left(\frac{n}{8}\right) + \left[\frac{9}{16}n^2 + \frac{3}{4}n^2 + n^2\right]$$

$$T\left(\frac{n}{2}\right) = \boxed{3T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2}$$

$$T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

After k iterations

$$= 3^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1}\left(\frac{3}{4}\right)^i n^2$$

Plug in k s.t. $\frac{n}{2^k} = 1 \Rightarrow k = \log_2 n$

$$= 3^{\log_2 n} T(1) + n^2 \sum_{i=0}^{k-1}\left(\frac{3}{4}\right)^i$$

$$\leq n^{\log_2 3} + n^2 \sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i$$

$$= n^{\log_2 3} + \frac{n^2}{1} \times \frac{1}{1 - \frac{3}{4}}$$

$$= n^{\log_2 3} + \underline{4n^2}$$

$$= O(n^2)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2 \quad , \quad T(1) = 1$$

$$= 2\left[2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$= 4T\left(\frac{n}{4}\right) + \left[\frac{n^2}{2} + n^2\right]$$

$$= 4\left[2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + \left[\frac{n^2}{2} + n^2\right]$$

$$= 8T\left(\frac{n}{8}\right) + \left[\frac{n^2}{4} + \frac{n^2}{2} + n^2\right]$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

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After k iterations

$$= 2^k T\left(\frac{n}{2^k}\right) + n^2 \sum_{i=0}^{k-1} \left(\frac{1}{2}\right)^i$$

Plug in value of k such that $\frac{n}{2^k} = 1 \Rightarrow 2^k = n$

$$\leq n\,T(1) + n^2 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= n + n^2\left(\frac{1}{1-\frac{1}{2}}\right)$$

$$= n + 2n^2$$

$$= O(n^2)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \quad , \quad T(1) = 1$$

$$= 4\left[4T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$$

$$= 16\,T\left(\frac{n}{4}\right) + \left[2n + n\right]$$

$$= 16\left[4T\left(\frac{n}{8}\right) + \frac{n}{4}\right] + \left[2n + n\right]$$

$$= 64\,T\left(\frac{n}{8}\right) + \left[4n + 2n + n\right]$$

$$T\left(\frac{n}{2}\right) = 4T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$T\left(\frac{n}{4}\right) = 4T\left(\frac{n}{8}\right) + \frac{n}{4}$$

After k iterations, we get

$$= 4^k\,T\left(\frac{n}{2^k}\right) + n\sum_{i=0}^{k-1} 2^i$$

Plug in value of k such that $\frac{n}{2^k} = 1 \Rightarrow 2^k = n$

$4^k = 2^{2k} = (2^k)^2 = n^2.$

$$= n^2\,T(1) + n \cdot \sum_{i=0}^{\log_2 n - 1} 2^i$$

$$= n^2 + n\left[\frac{2^{\log_2 n} - 1}{2 - 1}\right]$$

$$= n^2 + n(n-1)$$

$$= n^2 + n^2 - n$$

$$= O(n^2)$$

Master Theorem

$$T(n) = A\,T\left(\frac{n}{B}\right) + O(n^k)$$

if $\boxed{\log_B A < k}$, then $O(n^k)$

if $\log_B A = k$, then $O(n^k \lg n)$

if $\log_B A > k$, then $O(n^{\log_B A})$

$\rightarrow A < B^k$

$\rightarrow A = B^k$

$\rightarrow A > B^k$