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Alliances in graphs: Parameters, properties and applications—A survey

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Abstract

In practice, an alliance can be a bond or connection between individuals, families, states, or entities, etc. Formally, a non-empty set S of vertices of a graph G is a defensive k-alliance (resp. an offensive k-alliance) if every vertex of S (resp. the boundary of S) has at least k more neighbors inside of S than it has outside of S. A powerful k-alliance is both defensive k-alliance and offensive $(k+2)$-alliance. During the last decade there has been a remarkable development on these three kinds of alliances. Due to their variety of applications, the alliances in its broad sense have received a special attention from many scientists and researchers. There have been applications of alliances in several areas such as bioinformatics, distributed computing, web communities, social networks, data clustering, business, etc. Several k-alliance numbers have been defined and a huge number of theoretical (algorithmic and computational) results are obtained for various graph classes. In this paper, we present a survey which covers a number of practical applications of alliances and the vast mathematical properties of the three types of k-alliances by giving a special attention to the study of the associated k-alliance (partition) numbers for different graph classes.

Keywords: Defensive (offensive powerful) k-alliance; Boundary k-alliance; Partitioning of graphs; k-alliance (partition) number; Graph class

Contents

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1. Introduction

1.1. Historical view and applications of alliances

The word alliance can be defined as a union or association formed for mutual benefit, for example it can be: a formal agreement or treaty between two or more nations to cooperate for specific purposes, a merging of efforts or interests by persons, families, states, or organizations. The study of alliances in graphs is first investigated by Kristiansen et al. [1] by defining different types of alliances that have been extensively studied in the last decade. These types of alliances are called defensive alliances [2,3], offensive alliances [4,5] and dual or powerful alliances [6,7]. A generalization of these alliances called k-alliances (or r-alliances) introduced by Shafique and Dutton [8,9] have received a special attention in recent years. In this setting, there have been definitions of many and various parameters which have been studied widely in the case of defensive (offensive, powerful) k-alliances and for different graphs classes. The study of the alliances (in its broad sense) is motivated by interesting applications in several areas.

The research work of Kristiansen et al. [1] is motivated by the alliance of nations in war for mutual support. They considered two cases: the first which corresponds to a defensive alliance is realized when nations are obligated to join forces if one or more of them are attacked and the second is an offensive alliance which is used as a mean of keeping the peace; as an example of illustration, the action of NATO troops in a war-torn country is given. Thus in the corresponding graph of this situation, the vertices represent the nations and the edges correspond to possible relations (of either friendship or hostility) between them. Essentially, Kristiansen et al. [1] studied the mathematical properties of defensive alliances in graphs.

Haynes et al. [10] studied structural characteristics of a class of biomolecules in the bioinformatics arena by involving several graphical invariants based on domination numbers. In fact, they studied the applicability of graphs in the analysis of secondary RNA structure. They used graph-theoretic trees as a modeling method to represent secondary RNA motifs. Specifically, they utilized five domination parameters including the *global defensive* (-1)-*alliance number*, that are highly sensitive to the structural changes of small ordered trees, to identify which trees of orders seven and eight are RNA-like in structure. With this study, it is shown that graphical invariants, which aid in the optimization of computer and electrical networks, are useful and serve as an interesting tool for genomic and proteomic predictions.

In distributed computing, one of the central problems is how to deal with failures. Flocchini et al. [11], Peleg [12], Srimani and Xu [13], and Xu and Srimani [14] studied the fault tolerance of distributed computing and communication networks. Generally, a distributed system is represented by a graph where the vertices represent the processors and the edges correspond to different communications between them. Thus by using the process of local majority voting in graphs, processors are partitioned into two alliances. Furthermore, Srimani and Xu [13], and Xu and Srimani [14] designed self-stabilizing fault tolerant distributed algorithms for the global defensive (offensive) alliances in a given arbitrary graph.

Flake et al. [15] defined a community on the web as a set of web pages that link (in either direction) to more web pages in the community than to pages outside of the community. With this definition, the defensive alliances represent exactly the mathematical model of such web communities.

Szabö and Czárán [16] and Kim and Liu [17] have studied defensive alliances in cyclical interaction models of six mutating species which represent generalizations of the Rock–Scissors–Paper game.

The alliances are also used in business and social networks in order to achieve common objectives by partners. In this context, Dickson and Weaver [18] studied the interaction between the firm size and the level of national R&D intensity to determine if it would be interesting for an SME (small and medium enterprise) to form a strategic alliance. Furthermore, H. Chen and T.J. Chen [19] investigated the strategic alliances between organizations by providing empirical evidence to show what kind of resources should be shared and how such resource sharing should be organized between partners.

The partitioning of a set of objects is a process which partitions these objects into subsets, so that the objects of the same group have similar characteristics, and two objects belonging to two distinct subsets are dissimilar. In fact, this process is subjective because the same set of elements should often be divided differently for various applications. It is known that the partitioning process is involved in many applications occurred in various areas such as: clustering of data, load balancing in parallel machines, image segmentation, data mining, scientific computing, VLSI design, task scheduling, parallel programming, geographical information systems, division of space air, classification of documents, etc.

One of the approaches used to solve the partitioning problem is to reduce it to a problem of graphs where the objects are represented by vertices and the edges correspond to possible relationships between the objects. Although some problems try to partition the edges of a graph, we usually mean by the partition of a graph the partition of the vertices of this graph. Once the graph is obtained, the problem is to find a partition of the graph into subgraphs according to a certain criterion. However, the main question is which criterion to choose so that we obtain the "best partition"?

For example, the partition of a graph into the least number of independent sets is used to solve the chromatic number problem. On the other hand, there exist several studies on partitioning graphs into k-alliances and various theoretical results are obtained for some graphs classes in the literature. In this framework, the partitioning of graphs into defensive k-alliances is investigated by Eroh and Gera [20,21], Haynes and Lachniet [22], Yero et al. [23], the partitioning into offensive k-alliances is the subject matter of Sigarreta et al. [24], Yero [25], and the partitioning into powerful k-alliances is discussed in Slimani and Kheddouci [26], Yero and Rodríguez-Velázquez [27]. In this context, it would be interesting to see how such partitioning can be of interest to solve practical problems.

Shafique [28] studied the partitioning of data (objects) into clusters by involving the concept of defensive 0-alliance. In general, the clusters are defined by maximizing the similarities of objects belonging to each cluster as well as the dissimilarities of objects between the clusters. Thus, there exist more bonds inside each cluster than between the clusters. He represented this situation by a graph where the vertices correspond to data and the edges symbolize the common property (similarity) that the data share. This implies that the vertices in every cluster have at least as many edges adjacent to the vertices inside the cluster as to the vertices outside it. This corresponds to the definition of defensive 0-alliance. On the basis of this, Shafique [28] established an approximate algorithm and applied it for different clustering applications.

In mobile ad hoc networks (MANETs), ensure confidentiality and secure communications in groups is a critical task. In this context, Seba et al. [29] proposed a fully distributed and self-stabilizing clustering algorithm for key management in MANETs by using the concept of defensive (-1)-alliance as a clustering criterion. Thus, they proposed a solution which meets the criteria of self-organization and mobility resilience. With experiments, they showed that the concept of defensive (-1)-alliance is an efficient clustering criterion for group key management in MANETs by comparing to other existing clustering schemes.

Recently, there have been two surveys in the literature on alliances: the first by Yero and Rodríguez-Velázquez [3] presents essentially results on one type of k-alliances in graphs namely those are defensive, and in the second Fernau and Rodríguez-Velázquez [30] have investigated the problem of several graph parameters which are known under completely different names in different areas and they have proposed a new framework called (global) (D, O)-alliances in order to unify their notations.

In this paper, we survey the vast mathematical properties of defensive, offensive and powerful k-alliances in graphs by presenting a large number of theoretical results corresponding to bounds and/or exact values obtained for the associated k-alliance (partition) numbers for various graph classes. There is an originality in our draft by surveying the most important results by classifying them firstly according to the "*type of k-alliance*" then by the criterion "*graph class*". This allows us to see how the study of various k-alliance (partition) numbers varies according to the two criteria "graph class" and "type of k-alliance". In particular, it is deduced the most/least studied graph classes (type of k-alliances) on which there are more/less results. The paper is partitioned into three principal parts: the first part given in Section 2 is devoted to the study of defensive k-alliances, the second part given in Section 3 deals with the offensive k-alliances, and the third part given in Section 4 discusses the powerful k-alliances. At the end of each part, we establish a table that summarizes the principal results obtained for every k-alliance (partition) number according to the different graph classes and we discuss some of their relationships and properties. Finally, in Section 5 we summarize and draw conclusions.

1.2. Terminology and definitions

In this part, we give some terminology and definitions which will be heavily used in the rest of this paper. Let $G = (V, E)$ be an undirected finite graph without loops and multiple edges where V denotes the vertex set and E denotes the edge set with $|V| = n$ and $|E| = m$. For a non-empty subset $S \subseteq V$, $\langle S \rangle$ denotes the subgraph of G induced by S. For any vertex $v \in V$, $N(v)$ is the *open neighborhood* of the vertex v, *i.e.* the set of vertices that are adjacent to v in G, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The number $\deg_S(v) = |N_S(v)|$ is the degree of v in S with $N_S(v) = \{u \in S : uv \in E\}$ is the set of neighbors v has in S and $N_S[v] = N_S(v) \cup \{v\}$. The *open neighborhood of* S is $N(S) = \bigcup_{v \in S} N(v)$ and the *closed neighborhood of* S is $N[S] = N(S) \cup S$. The *boundary* of S is the set $\partial S = N(S) - S$ and the complement of S in V is $\bar{S} = V - S$. We denote the degree sequence of G by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Other notations will be introduced when needed.

A set S is a *dominating set* if $N[S] = V$ and it is a *total dominating set* or an *open dominating set* if $N(S) = V$. The minimum cardinality of a dominating set (resp. total dominating set) of G is the *domination number* $\gamma(G)$ (resp. *total domination number* $\gamma_t(G)$).

A non-empty set of vertices $S \subseteq V$ is called *defensive alliance* if for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap \bar{S}|$ or equivalently $\deg_S(v) + 1 \geq \deg_{\bar{S}}(v)$. In this case, we say that every vertex in S is *defended* from possible attack by vertices in $\bar{S}$. A defensive alliance S is called *strong* if for every vertex $v \in S$, $\deg_S(v) + 1 > \deg_{\bar{S}}(v)$. A set $S \subset V$ is called *offensive alliance*, if for every vertex $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap \bar{S}|$ or equivalently $\deg_S(v) \geq \deg_{\bar{S}}(v) + 1$. In this case, we say that every vertex in $\partial(S)$ is *vulnerable* to possible attack by vertices in S. An offensive alliance S is called *strong* if for every vertex $v \in \partial(S)$, $\deg_S(v) > \deg_{\bar{S}}(v) + 1$. The alliances that are both defensive and offensive are called *powerful alliances*. That is, $S \subset V$ is a powerful alliance if for every vertex $v \in N[S]$, $|N[v] \cap S| \geq |N[v] \cap \bar{S}|$. An alliance S of any type (defensive, offensive or powerful) is called *global* if S is a dominating set, and it is called *critical or minimal* if no proper subset of S is an alliance of the same type. See the graphs given in Figs. 1(a)–1(c) of Appendix for illustration.

A subset $S \subseteq V$ is a defensive k-alliance, with $k \in \{-\delta_1, \ldots, \delta_1\}$, if for every $v \in S$, $\deg_S(v) \geq \deg_{\bar{S}}(v) + k$. A set $S \subset V$ is an offensive k-alliance, with $k \in \{2 - \delta_1, \ldots, \delta_1\}$ if for every $v \in \partial(S)$, $\deg_S(v) \geq \deg_{\bar{S}}(v) + k$. A set $S \subset V$ is a powerful k-alliance if it is both a defensive k-alliance and an offensive $(k+2)$-alliance. Yero and Rodríguez-Velázquez [31,32] studied the limit case of defensive (resp. offensive and powerful) k-alliances and by considering equalities in their associated definitions they defined the so-called boundary defensive (resp. offensive and powerful) k-alliances. See the graphs given in Figs. 1(d)–1(f) of Appendix for illustration.

2. Defensive k-alliances in graphs

In this section, we study mathematical properties of defensive k-alliances by giving bounds and/or exact values of several parameters studied for various graphs classes. A defensive k-alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that every vertex in S has at least k more neighbors in S than it has outside of S. The case $k = -1$ (resp. $k = 0$) corresponds to the standard *defensive alliances* (resp. *strong defensive alliances* which is also known as a *cohesive set*) defined in [1].

Several parameters have been defined and studied in the literature for defensive k-alliances, one can see [1,2,33–36] and others. These parameters are defined as follows: The *defensive* (-1)*-alliance number* known as *defensive alliance number* $a^d_{-1}(G)$ (resp. *defensive* 0*-alliance number* known as *strong defensive alliance number* $a^d_0(G)$) is the minimum cardinality among all (critical) defensive (-1)-alliances (resp. defensive 0-alliances) of G [1]. The *global defensive* (-1)*-alliance number* $\gamma^d_{-1}(G)$ (resp. *global defensive* 0*-alliance number* $\gamma^d_0(G)$) is the minimum cardinality among all (critical) global defensive (-1)-alliances (resp. global defensive 0-alliances) of G [2]. The *upper defensive* (-1)*-alliance number* $A^d_{-1}(G)$ (resp. *upper defensive* 0*-alliance number* $A^d_0(G)$) is the maximum cardinality among all critical defensive (-1)-alliances (resp. defensive 0-alliances) of G [1,28]. The *defensive k-alliance number* $a^d_k(G)$ is the minimum cardinality among all (critical) defensive k-alliances of G [9,28]. The *global defensive k-alliance number* $\gamma^d_k(G)$ is the minimum cardinality among all (critical) global defensive k-alliances of G [37]. The *upper defensive k-alliance number* $A^d_k(G)$ is the maximum cardinality among all critical defensive k-alliances of G [28].

Now, we give some basic relations and observations which bind various parameters of defensive k-alliances. For any graph $G = (V, E)$, we have:

(1) $a^d_{-1}(G) \leq a^d_0(G) \leq A^d_0(G)$ [1,28];
(2) $a^d_{-1}(G) \leq A^d_{-1}(G)$ [1,28];
(3) $a^d_{-1}(G) \leq \gamma^d_{-1}(G)$ [2,38];
(4) $a^d_0(G) \leq \gamma^d_0(G)$ [28,38];
(5) $a^d_{-1}(G) = 1 \Leftrightarrow \exists\, v \in V,\ \deg(v) \leq 1$ [1];
(6) $a^d_0(G) = 1 \Leftrightarrow G$ contains an isolated vertex [1];
(7) $a^d_{-1}(G) = 2 \Leftrightarrow \delta_n \geq 2$ and G has two adjacent vertices of degree at most 3 [1];
(8) $a^d_0(G) = 2 \Leftrightarrow \delta_n \geq 1$ and G has two adjacent vertices of degree at most 2 [1];
(9) $a^d_{-1}(G) \leq a^d_0(G) \leq \gamma^d_0(G)$ [2,38];
(10) $\gamma(G) \leq \gamma^d_{-1}(G) \leq \gamma^d_0(G)$ [2,38,39];
(11) $a^d_k(G) \leq a^d_{k+1}(G)$ [23,34,37];
(12) $a^d_k(G) \leq \gamma^d_k(G)$ [23,37];
(13) $\gamma(G) \leq \gamma^d_k(G) \leq \gamma^d_{k+1}(G)$ [23,37];
(14) $A^d_k(G) \leq A^d_{k+1}(G)$ [35].

2.1. Study of defensive k-alliance numbers for various graphs classes

Defensive k-alliances are extensively studied in the literature for different graphs classes. In this subsection, we present important theoretical results obtained for this type of alliance. Essentially, we give bounds or exact values established for defensive k-alliance numbers studied for various graphs classes.

2.1.1. General graphs

We present essential results concerning defensive k-alliances in the case of general graphs. In particular, we give bounds obtained for various defensive k-alliance numbers by using different graph parameters. Let $G = (V, E)$ be a general graph of order n.

The study of defensive k-alliances in graphs was introduced by Kristiansen et al. [1]. They proposed some sharp bounds for the *defensive* (-1)-*alliance number* and the *defensive* 0-*alliance number* as follows: for any connected graph G, $a^d_{-1}(G) \leq \min\{n - \lceil \frac{\delta_n}{2} \rceil, \lceil \frac{n}{2} \rceil\}$ and $a^d_0(G) \leq \min\{n - \lfloor \frac{\delta_n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}$ (note that these bounds are attained, for example, for the complete graph $G = K_n$). Rodríguez-Velázquez and Sigarreta [40] studied the relationship between the (global) defensive k-alliance numbers of a graph, its algebraic connectivity (the second smallest eigenvalue of the Laplacian matrix of the graph G) and its spectral radius (the largest eigenvalue of the adjacency matrix of the graph G). They obtained lower bounds for the parameters $a^d_{-1}(G)$ and $a^d_0(G)$ by using the maximum degree δ_1 and the algebraic connectivity μ. Thus they showed that for a simple graph G, $a^d_{-1}(G) \geq \left\lceil \frac{n\mu}{n+\mu} \right\rceil$, $a^d_0(G) \geq \left\lceil \frac{n(\mu+1)}{n+\mu} \right\rceil$ and for a simple connected graph G, $a^d_0(G) \geq \left\lceil \frac{n(\mu - \lfloor \frac{\delta_1}{2} \rfloor)}{\mu} \right\rceil$ (note that this latter bound is reached, for example, in the following cases given in [40]: the complete graph, the Petersen graph, and the 3-cube graph). Other bounds for the same parameters are given by Araujo-Pardo and Barrière [41] by using the minimum degree of the graph and its girth.

Haynes et al. [2] investigated the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number*. They obtained sharp bounds and showed that: if G is a graph of order n, then $\gamma^d_{-1}(G) \geq \frac{\sqrt{4n+1}-1}{2}$ and $\gamma^d_0(G) \geq \sqrt{n}$; and for any graph G with no isolated vertices and minimum degree δ_n, $\gamma^d_{-1}(G) \leq n - \lceil \frac{\delta_n}{2} \rceil$ and $\gamma^d_0(G) \leq n - \lfloor \frac{\delta_n}{2} \rfloor$. They also obtained that if G is a graph of order n, then $\gamma^d_{-1}(G) \geq \frac{n}{\lceil \frac{r}{2} \rceil + 1}$. Moreover, Haynes et al. [2] obtained other results for the same parameters by using the total domination number $\gamma_t(G)$. Thus, for the *global defensive* (-1)-*alliance number* they showed that for any graph G with $\delta_n \geq 2$, $\gamma^d_{-1}(G) \geq \gamma_t(G)$, and furthermore if $\delta_1 \leq 3$ then $\gamma^d_{-1}(G) = \gamma_t(G)$. For the *global defensive* 0-*alliance number* they showed that for any graph G with no isolated vertices, $\gamma^d_0(G) \geq \gamma_t(G)$. On the other hand, Rodríguez-Velázquez and Sigarreta [40] gave lower bounds for these parameters in terms of the order of a simple graph G, its maximum degree δ_1 and its spectral radius λ. These results are: $\gamma^d_{-1}(G) \geq \lceil \frac{n}{\lambda+2} \rceil$, $\gamma^d_{-1}(G) \geq \left\lceil \frac{2n}{\delta_1+3} \right\rceil$, $\gamma^d_0(G) \geq \lceil \frac{n}{\lambda+1} \rceil$ and $\gamma^d_0(G) \geq \left\lceil \frac{n}{\lfloor \frac{\delta_1}{2} \rfloor + 1} \right\rceil$. For more bounds on these parameters, one can see Favaron [39], Hsua et al. [42] and Sigarreta and Rodríguez-Velázquez [36], where other concepts such as the minimum cardinality of an independent set, the dominating number, and the diameter of graph G are used.

Yero et al. [23] presented some relations for the *(global) defensive k-alliance number* by considering the cases where the degrees of vertices and k are even/odd. Thus, they obtained that if every vertex of G has even degree and k is odd, $k = 2l - 1$, then every (global) defensive $(2l - 1)$-alliance in G is a (global) defensive $(2l)$-alliance and vice versa. Hence, in such a case, $a^d_{2l-1}(G) = a^d_{2l}(G)$ and $\gamma^d_{2l-1}(G) = \gamma^d_{2l}(G)$. Analogously, if every vertex of G has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in G is a defensive $(2l + 1)$-alliance and vice versa. Hence, in such a case, $a^d_{2l}(G) = a^d_{2l+1}(G)$ and $\gamma^d_{2l}(G) = \gamma^d_{2l+1}(G)$. Rodríguez-Velázquez et al. [34] and Sigarreta in his thesis [43] studied the defensive k-alliances and showed that for every $k \in \{-\delta_n, ..., \delta_1\}$, the *defensive k-alliance number* satisfies $\lceil \frac{\delta_n+k+2}{2} \rceil \leq a^d_k(G) \leq n - \lfloor \frac{\delta_n - k}{2} \rfloor$, and if $k \in \{-\delta_n, \ldots, 0\}$ one has the upper bound $a^d_k(G) \leq \lceil \frac{n+k+1}{2} \rceil$ (note that these bounds are attained, for example, for the complete graph $G = K_n$ for every $k \in \{1 - n, \ldots, n - 1\}$). Moreover, for every $k, r \in \mathbb{Z}$ such that $-\delta_n \leq k \leq \delta_1$ and $0 \leq r \leq \frac{k+\delta_n}{2}$, they showed that $a^d_{k-2r}(G) + r \leq a^d_k(G)$. They also gave other bounds by involving the algebraic connectivity of G. Thus, for any connected graph G and $k \in \{-\delta_n, \ldots, \delta_1\}$, $a^d_k(G) \geq \lceil \frac{n(\mu+k+1)}{n+\mu} \rceil$ and $a^d_k(G) \geq \left\lceil \frac{n(\mu - \lfloor \frac{\delta_1 - k}{2} \rfloor)}{\mu} \right\rceil$. By using the isoperimetric number $\mathcal{I}(G)$ and the algebraic connectivity μ, Yero [25] and Yero et al. [23] obtained other bounds for the same parameter. They showed that for any graph G if it is partitionable into defensive k-alliances, then

$a_k^d(G) \geq \mathcal{I}(G) + k + 1$ and $a_k^d(G) \geq \left\lceil \frac{\mu+2(k+1)}{2} \right\rceil$. Note that this latter bound is reached for example for the graph $G = C_3 \times C_3$ for $k = 0$, given in [23,25], in this case $\mu = 3$. We recall that the isoperimetric number of a graph $G = (V, E)$ is defined as $\mathcal{I}(G) = \min_{S \subset V : |S| \leq \frac{n}{2}} \left\{ \frac{\sum_{v \in S} \deg_{\bar{S}}(v)}{|S|} \right\}$ or $\mathcal{I}(G) = \min \frac{|E(X,Y)|}{\min\{|X|,|Y|\}}$ with $X, Y \subseteq V$.

Fernau et al. [44] studied the global defensive k-alliances and established bounds for the *global defensive k-alliance number*. Thus they presented that for any graph G, $\frac{\sqrt{4n+k^2}+k}{2} \leq \gamma_k^d(G) \leq n - \lceil \frac{\delta_n - k}{2} \rceil$ and $\gamma_k^d(G) \geq \left\lceil \frac{n}{\left\lfloor \frac{\delta_1 - k}{2} \right\rfloor + 1} \right\rceil$ (note that these bounds are attained, for instance, for the graph given in Fig. 1(g) of Appendix where $\gamma_{-4}^d(G) = 1$). These results are also obtained by Rodríguez-Velázquez et al. [37] and Sigarreta [43] by showing that these bounds are a generalization of those obtained by Haynes et al. [2] for $\gamma_{-1}^d(G)$ and $\gamma_0^d(G)$ in the cases of general and bipartite graphs. Furthermore, Rodríguez-Velázquez et al. [37] and Sigarreta [43] showed that for S a global defensive k-alliance of minimum cardinality in G, if $W \subset S$ is a dominating set in G then for every $r \in \mathbb{Z}$ such that $0 \leq r \leq \gamma_k^d(G) - |W|$, $\gamma_{k-2r}^d(G) + r \leq \gamma_k^d(G)$. By using the spectral radius λ, Sigarreta [43] also obtained that for every graph G, $\gamma_k^d(G) \geq \lceil \frac{n}{\lambda - k + 1} \rceil$.

The upper defensive k-alliances have been studied by Sigarreta [35]. He established a bound for the *upper defensive k-alliance number* in terms of the order of G and its minimum degree. Thus for every $k \in \{-\delta_n, \ldots, \delta_1\}$ and for every graph G, $A_k^d(G) \leq \lceil \frac{2n - \delta_n + k}{2} \rceil$; and if every $S \subset V$ such that $|S| \geq r$ is a defensive k-alliance, then $A_k^d(G) \leq r$. He also gave an other upper bound by defining and using a new concept $\phi_k^d(G)$ which is the largest cardinality of a maximal defensive k-alliance free set. A set $X \subset V$ is defensive k-alliance free, if for all defensive k-alliance S, $S \setminus X \neq \emptyset$ (X does not contain any defensive k-alliance as a subset). A defensive k-alliance free set X is maximal if for every $v \notin X$, there exists $S \subseteq X$ such that $S \cup \{v\}$ is a defensive k-alliance. Thus, for every $k \in \{-\delta_1, \ldots, \delta_1\}$ one has $A_k^d(G) \leq \phi_k^d(G) + 1$.

By considering the limit case of k-alliances, Yero in his thesis [25] and Yero and Rodríguez-Velázquez [31] defined a new variant of k-alliances called boundary k-alliances. They studied mathematical properties of such alliances by obtaining in particular several bounds on the cardinality of every boundary defensive k-alliance. Thus, if S is a boundary defensive k-alliance in a graph G, then $\lceil \frac{\delta_n + k + 2}{2} \rceil \leq |S| \leq \lfloor \frac{2n - \delta_n + k}{2} \rfloor$ (note that these two bounds are reached, for instance, for the complete graph $G = K_n$ for every $k \in \{1 - n, \ldots, n - 1\}$). Furthermore, for a connected graph G, if S is a boundary defensive k-alliance in G then $\left\lceil \frac{n(\mu - \lfloor \frac{\delta_1 - k}{2} \rfloor)}{\mu} \right\rceil \leq |S| \leq \left\lfloor \frac{n(\mu_* - \lceil \frac{\delta_n - k}{2} \rceil)}{\mu_*} \right\rfloor$ and $\left\lceil \frac{n(\mu + k + 2) - \mu}{2n} \right\rceil \leq |S| \leq n - \left\lceil \frac{n(\mu - k) - \mu}{2n} \right\rceil$, where μ_* is the Laplacian spectral radius (the largest Laplacian eigenvalue of the graph G).

2.1.2. Tree graphs

By definition, a tree $T = (V, E)$ of order n and size m is a connected graph with $m = n - 1$. The study of trees is particularly interesting because various applications are modeled and solved by using their properties. Thus, the study of alliances in general and defensive alliances in particular for this class of graph is important. In what follows, we present some results concerning defensive k-alliance numbers in trees.

Kristiansen et al. [1] studied the defensive k-alliances and gave an exact value for the *defensive (-1)-alliance number* and an upper bound for the *defensive 0-alliance number*. These results are: $a_{-1}^d(T) = 1$ and $a_0^d(T) \leq n$.

Haynes et al. [2] obtained upper bounds and sharp lower bounds for the *global defensive (-1)-alliance number* and the *global defensive 0-alliance number* as follows: if T is a tree of order n, then $\gamma_{-1}^d(T) \geq \frac{n+2}{4}$, $\gamma_0^d(T) \geq \frac{n+2}{3}$, $\gamma_{-1}^d(T) \leq \frac{3n}{5}$ for $n \geq 4$ (with equality for the latter bound if and only if $T \in \mathcal{T}_1$ with $\mathcal{T}_1$ is a special family of trees [2]), and $\gamma_0^d(T) \leq \frac{3n}{4}$ for $n \geq 3$ (with equality if and only if T belongs to a special family of trees $\mathcal{T}_2$ [2]). Rodríguez-Velázquez and Sigarreta [45] gave more general lower bounds for the same parameters by imposing a condition on the number of connected components of the subgraphs induced by the alliances. They showed that if S is a global defensive (-1)-alliance (resp. 0-alliance) in a tree T such that the subgraph $\langle S \rangle$ has c connected components, then $|S| \geq \lceil \frac{n+2c}{4} \rceil$ (resp. $|S| \geq \lceil \frac{n+2c}{3} \rceil$). As a particular case of these results, if $\langle S \rangle$ is connected, they obtained lower bounds for $\gamma_{-1}^d(T)$ and $\gamma_0^d(T)$ already proved by Haynes et al. in [2]. On the other hand, Chen and Chee [46] proved that for a tree T of order $n \geq 3$ having s support vertices, $\gamma_{-1}^d(T) \leq \frac{n+s}{2}$, with equality if and only if T belongs to a special family of trees ξ [46] (we recall that a vertex of degree one is called a leaf and its neighbor is a support vertex). Bouzefrane et al. [47] showed that if T is a tree of order $n \geq 2$ with l leaves and s support vertices, then

$\gamma_{-1}^{d}(T) \geq \frac{3n-l-s+4}{8}$ (with equality if and only if $T = P_2$ or $T \in \mathcal{T}$ with $\mathcal{T}$ is a special family of trees [47]) and $\gamma_0^{d}(T) \geq \frac{3n-l-s+4}{6}$ (with equality if and only if T belongs to a special family of trees $\mathcal{F}$ [47]).

Harutyunyan [48] studied the *global defensive* (-1)-*alliance number* for the complete t-ary tree $T_{t,d}$ and for its particular case the complete binary tree $T_d = T_{2,d}$. A t-ary tree is a rooted tree where each vertex has at most t children. A complete t-ary tree is a t-ary tree in which all the leaves have the same depth and all the vertices except the leaves have t children; thus $T_{t,d}$ is the complete t-ary tree with depth d. For a complete binary tree T_d of order n, Harutyunyan gave an exact value for the *global defensive* (-1)-*alliance number*: $\gamma_{-1}^{d}(T_d) = \gamma_{-1}^{d}(T_{2,d}) = \left\lceil \frac{2n}{5} \right\rceil$ for any d. For complete t-ary tree $T_{t,d}$, he obtained lower and upper bounds for the same parameter by means of t and d. Thus, for $d \geq 2$ and $t \geq 2$, we have $t^{d-1}\left\lfloor \frac{t-1}{2} \right\rfloor + t^{d-1} + t^{d-2} \leq \gamma_{-1}^{d}(T_{t,d}) \leq t^{d-1}\left\lfloor \frac{t-1}{2} \right\rfloor + t^{d-1} + t^{d-2} + t^{d-3}$. Moreover, exact values for $T_{t,d}$, $t = 3$ and $t = 4$ are given in [48,49]. On the other hand, for any tree T of order n Harutyunyan presented a bound for $\gamma_{-1}^{d}(T)$ by using the *global offensive* (-1)-*alliance number* $\gamma_{-1}^{o}(T)$ (that we will study in the next section). This result is $\gamma_{-1}^{d}(T) \leq \gamma_1^{o}(T) + \frac{n}{2}$. By using the independence number $\beta(T)$ (the maximum cardinality of an independent set in T), Chellali and Haynes [50] gave sharp bounds for the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number* as follows: for any tree T, $\gamma_{-1}^{d}(T) \leq \beta(T)$, furthermore they obtained that for every nontrivial tree T with l leaves $\gamma_{-1}^{d}(T) \leq \frac{n+l-1}{2}$; note that for $l \leq \frac{n}{5}$ this latter bound is an improvement of the one ($\gamma_{-1}^{d}(T) \leq \frac{3n}{5}$ for $n \geq 4$) given by Haynes et al. [2]. For the *global defensive* 0-*alliance number* they showed that if T is a tree of order $n \geq 3$ with s support vertices, then $\gamma_0^{d}(T) \leq \frac{3\beta(T)-1}{2}$ and $\gamma_0^{d}(T) \leq \beta(T) + s - 1$.

Favaron [39] compared the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number* to the independent domination number i. He obtained bounds in the forms $i(T) \leq f(\gamma_{-1}^{d}(T))$ and $i(T) \leq g(\gamma_0^{d}(T))$ for special families of trees, where f and g are functions.

Rodríguez-Velázquez and Sigarreta [37] and Sigarreta [43] considered global defensive k-alliances and they obtained a lower bound for the cardinality of every global defensive k-alliance in trees, by imposing a condition on the number of connected components of the subgraphs induced by the k-alliances. Thus, if S is a global defensive k-alliance in T such that the subgraph $\langle S \rangle$ has c connected components, then $|S| \geq \left\lceil \frac{n+2c}{3-k} \right\rceil$ (note that the authors gave two unusual graphs for which this bound is reached). As a particular case of this result, if $\langle S \rangle$ is connected, Rodríguez-Velázquez and Sigarreta [37] and Sigarreta [43] obtained lower bounds for $\gamma_{-1}^{d}(T)$ and $\gamma_0^{d}(T)$ already proved by Haynes et al. in [2]. Furthermore, as a consequence of this same result, they obtained that for every tree T of order n, $\gamma_k^{d}(T) \geq \left\lceil \frac{n+2}{3-k} \right\rceil$. This latter bound is attained for $k \in \{-4, -3, -2, 0, 1\}$ in the case of $G = K_{1,4}$ as given in [37,43].

2.1.3. Planar graphs

We say that a graph is planar if one can draw it in the plan so that its edges do not cross. In this paragraph, we put on view the essential results obtained on defensive k-alliance parameters for this type of graphs. Let $P = (V, E)$ be a planar graph of order n.

A global alliance S is said to be an empire if with respect to a planar embedding of G, each connected component of $\langle S \rangle$ can be enclosed by a closed Jordan curve — a "wall" surrounding a fortress, where the region outside of each Jordan curve contains all vertices of $\bar{S}$. Enciso and Dutton [33] and Enciso [51] used this concept and showed that for a planar graph P where $S \subseteq V$ is a global defensive (-1)-alliance (resp. global defensive 0-alliance) of P, if S is an empire then $|S| \geq \lceil \frac{n+6}{6} \rceil$ (resp. $|S| \geq \lceil \frac{n+6}{5} \rceil$). Rodríguez-Velázquez and Sigarreta [45] presented tight bounds in planar graphs for the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number* according to the order n as follows:

(i) If $n > 6$, then $\gamma_{-1}^{d}(P) \geq \left\lceil \frac{n+12}{8} \right\rceil$.
(ii) If $n > 6$ and P is a triangle-free graph, then $\gamma_{-1}^{d}(P) \geq \left\lceil \frac{n+8}{6} \right\rceil$.
(iii) If $n > 4$, then $\gamma_0^{d}(P) \geq \left\lceil \frac{n+12}{7} \right\rceil$.
(iv) If $n > 4$ and P is a triangle-free graph, then $\gamma_0^{d}(P) \geq \left\lceil \frac{n+8}{5} \right\rceil$.

Furthermore, they proved that if S is a global defensive (-1)-alliance in a general graph G such that the subgraph $\langle S \rangle$ is planar connected with f faces, then $|S| \geq \lceil \frac{n-2f+4}{4} \rceil$ and in the case where S is a global defensive 0-alliance then $|S| \geq \left\lceil \frac{n-2f+4}{3} \right\rceil$. Rodríguez-Velázquez and Sigarreta [45] also showed that for a general graph G of order n where S is a global defensive (-1)-alliance such that $|S| \geq 2$, if $\langle S \rangle$ is planar and its minimum degree is at least σ, then

$|S| \geq \left\lceil \frac{\sigma-7+\sqrt{(\sigma-7)^2+4(12+n)}}{2} \right\rceil$; moreover, if $\langle S \rangle$ is also a triangle-free graph, then $|S| \geq \left\lceil \frac{\sigma-5+\sqrt{(\sigma-5)^2+4(8+n)}}{2} \right\rceil$. Note that, they presented several examples of graphs for which all the above bounds are reached.

On the other hand, Rodríguez-Velázquez and Sigarreta [37] and Sigarreta [43] presented tight lower bounds for the *global defensive k-alliance number*. They showed that for any planar graph P of order n:

(i) If $n > 2(2-k)$, then $\gamma_k^d(P) \geq \left\lceil \frac{n+12}{7-k} \right\rceil$.
(ii) If $n > 2(2-k)$ and P is a triangle-free graph, then $\gamma_k^d(P) \geq \left\lceil \frac{n+8}{5-k} \right\rceil$.

They also obtained that if a simple general graph G has a global defensive k-alliance S such that the subgraph $\langle S \rangle$ is planar connected with f faces, then $|S| \geq \left\lceil \frac{n-2f+4}{3-k} \right\rceil$. Note that, Rodríguez-Velázquez and Sigarreta [37] and Sigarreta [43] presented examples of graphs where all these bounds are attained.

Yero [25] and Yero and Rodríguez-Velázquez [31] studied the boundary defensive k-alliances. They showed that if S is a boundary defensive k-alliance in a general graph G such that $\langle S \rangle$ is planar connected with f faces then $|S| = \frac{\mathcal{C}+4-2f}{2-k}$ for $k \neq 2$, and $|S| \leq \left\lfloor \frac{\sqrt{16-8f+(n+k-2)^2}+n+k-2}{2} \right\rfloor$, where $\mathcal{C}$ is the number of edges of G with one endpoint in S and the other endpoint outside of S. Note that this latter bound is tight and it is attained for example for the complete graph $G = K_5$ where any subset S of G of cardinality four is a boundary defensive 2-alliance and $\langle S \rangle \cong K_4$ as given in [25,31]. Furthermore, they presented lower and upper bounds for $|S|$ according to the value of k. Thus, they proved that for a boundary defensive k-alliance S in a general graph G such that $\langle S \rangle$ is planar connected with $f > 2$ faces, if $k \in \{5-\delta_1, \dots, \delta_1\}$ (resp. $k \in \{5-\delta_n, \dots, \delta_1\}$) then $|S| \geq \lceil \frac{4f-8}{\delta_1+k-4} \rceil$ (resp. $|S| \leq \lfloor \frac{4f-8}{\delta_n+k-4} \rfloor$).

2.1.4. Complete graphs

Let $K_n = (V, E)$ be a complete graph of order n. In this part, we exhibit some exact values obtained for defensive k-alliance numbers in complete graphs.

Kristiansen et al. [1] investigated the defensive k-alliances and obtained exact values for the *defensive* (-1)*-alliance number* and the *defensive* 0*-alliance number* as follows: $a_{-1}^d(K_n) = \left\lceil \frac{n}{2} \right\rceil$ and $a_0^d(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Haynes et al. [2] studied the global defensive k-alliances and established exact values for the *global defensive* (-1)*-alliance number* and the *global defensive* 0*-alliance number*. These results are: $\gamma_{-1}^d(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ and $\gamma_0^d(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Rodríguez-Velázquez and Sigarreta [37], Rodríguez-Velázquez et al. [34] and Sigarreta [43] considered the (global) defensive k-alliances and obtained an exact value for the *(global) defensive k-alliance number*. Thus, they showed that for every $k \in \{1-n, \dots, n-1\}$, $a_k^d(K_n) = \gamma_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$. Sigarreta [35] studied the *upper defensive k-alliance number* and obtained the same value for this parameter as well, *i.e.* $A_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$.

Yero [25] and Yero and Rodríguez-Velázquez [31] studied the boundary defensive k-alliances and proved that the cardinality of every boundary defensive k-alliance S in the complete graph is $|S| = \frac{n+k+1}{2}$.

2.1.5. Bipartite graphs and complete bipartite graphs

A graph is bipartite if its vertices can be divided into two sets X and Y so that every edge of the graph connects a vertex in X to a vertex in Y. Let $B = (X, Y, E)$ be a bipartite graph of order n. $K_{r,s}$ is the complete bipartite graph where r (resp. s) is the cardinality of the set X (resp. Y).

Kristiansen et al. [1] studied the defensive k-alliances and established exact values for the *defensive* (-1)*-alliance number* and the *defensive* 0*-alliance number* in complete bipartite graphs. So they obtained that for $2 \leq r \leq s$, $a_{-1}^d(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$ and $a_0^d(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil$.

Haynes et al. [2] investigated the global defensive k-alliances and obtained sharp lower bounds for the *global defensive* (-1)*-alliance number* and the *global defensive* 0*-alliance number* in bipartite graphs $\gamma_{-1}^d(B) \geq \frac{2n}{\delta_1+3}$ and $\gamma_0^d(B) \geq \frac{2n}{\delta_1+2}$. Furthermore, they presented exact values for the same parameters in complete bipartite graphs as follows: $\gamma_{-1}^d(K_{1,s}) = \left\lfloor \frac{s}{2} \right\rfloor + 1$, $\gamma_{-1}^d(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$ if $r,\ s \geq 2$, and $\gamma_0^d(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil$. Favaron [39] compared the *global defensive* (-1)*-alliance number* and the *global defensive* 0*-alliance number* to the independent domination number i. He obtained bounds in the forms $i(B) \leq f(\gamma_{-1}^d(B))$ and $i(B) \leq g(\gamma_0^d(B))$ for special families of bipartite graphs, where f and g are functions.

Sigarreta [35] studied the mathematical properties of upper defensive k-alliances in graphs and presented some results for the *upper defensive k-alliance number* in complete bipartite graph $K_{r,s}$ where $r \geq s$. Thus, he proved that: $A_k^d(K_{r,s}) = 1$, for $k \in \{-r, \dots, -s\}$; $A_k^d(K_{r,s}) = \lceil \frac{r+k}{2} \rceil + \lceil \frac{s+k}{2} \rceil$, for $k \in \{-1, 0, \dots, s-1\}$; and $A_k^d(K_{r,s}) = r + s - \lfloor \frac{r-k}{2} \rfloor$, for $k \in \{s, \dots, r-1\}$.

2.1.6. Regular graphs

A graph of which all the vertices have the same degree is known as regular and if the common degree is δ then it is said that the graph is δ-regular. In this paragraph, we give some bounds or exact values obtained for defensive k-alliance numbers concerning this class of graphs. For this, we denote by $R_\delta = (V, E)$ the δ-regular graph of order n.

In the literature, the alliance numbers of δ-regular graphs are known only for small degrees. Kristiansen et al. [1] and Araujo-Pardo and Barrière [41] presented some exact values for the *defensive* (-1)-*alliance number*, the *defensive* 0-*alliance number*, the *upper defensive* (-1)-*alliance number* and the *upper defensive* 0-*alliance number* in δ-regular graphs. According to the value of δ and by using the concept of girth (the length of a smallest cycle in a graph) and $lc(R_\delta)$ (the maximum length of an induced cycle in the graph), they obtained the results given as follows:

(i) $a_{-1}^d(R_1) = A_{-1}^d(R_1) = 1$ and $a_0^d(R_1) = A_0^d(R_1) = 2$.
(ii) $a_{-1}^d(R_2) = A_{-1}^d(R_2) = a_0^d(R_2) = A_0^d(R_2) = 2$.
(iii) $a_{-1}^d(R_3) = A_{-1}^d(R_3) = 2$, $a_0^d(R_3) = girth(R_3)$ and $A_0^d(R_3) = lc(R_3)$.
(iv) $a_{-1}^d(R_4) = a_0^d(R_4) = girth(R_4)$ and $A_{-1}^d(R_4) = A_0^d(R_4) = lc(R_4)$.
(v) $a_{-1}^d(R_5) = girth(R_5)$ and $A_{-1}^d(R_5) = lc(R_5)$.

Haynes et al. [2] studied the global defensive k-alliances and gave a lower bound for the *global defensive* (-1)-*alliance number* in 4-regular graph that is $\gamma_{-1}^d(R_4) \geq \frac{n}{3}$. Note that this bound is also true for cubic graphs as mentioned in [2].

Yero [25] and Yero and Rodríguez-Velázquez [31] studied the boundary defensive k-alliances. They showed that for $k \in \{5 - \delta, \dots, \delta\}$, if S is a boundary defensive k-alliance in δ-regular graph such that $\langle S \rangle$ is planar connected with f faces, then $|S| = \frac{4f-8}{\delta+k-4}$, and $\mathcal{C} = \frac{2(\delta-k)(f-2)}{\delta+k-4}$, where $\mathcal{C}$ is the number of edges of the graph with one endpoint in S and the other endpoint outside of S.

2.1.7. Cycle graphs

In this paragraph, we exhibit exact values obtained for defensive k-alliance numbers for this class of graphs. Let $C_n = (V, E)$ be a cycle graph of order n.

Kristiansen et al. [1] investigated the defensive k-alliances and showed that the different defensive k-alliance numbers, the *defensive* (-1)-*alliance number*, the *defensive* 0-*alliance number*, the *upper defensive* (-1)-*alliance number* and the *upper defensive* 0-*alliance number*, have the same exact value which is equal to 2. Thus, $a_{-1}^d(C_n) = a_0^d(C_n) = A_{-1}^d(C_n) = A_0^d(C_n) = 2$.

Haynes et al. [2] studied the global defensive k-alliances and proved that in a cycle graph of order $n \geq 3$ the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number* are equal to the total domination number. Thus, $\gamma_{-1}^d(C_n) = \gamma_0^d(C_n) = \gamma_t(C_n)$.

2.1.8. Path graphs

Let $P_n = (V, E)$ be a path graph of order n. Kristiansen et al. [1] showed that for any path graph P_n, the *defensive* (-1)-*alliance number* satisfies $a_{-1}^d(P_n) = 1$ and the *defensive* 0-*alliance number* verify $a_0^d(P_n) = 2$. They also proved that for every path P_n with $n \geq 4$ the *upper defensive* (-1)-*alliance number* and the *upper defensive* 0-*alliance number* are equal to the same value. Thus, $A_{-1}^d(P_n) = A_0^d(P_n) = 2$ for $n \geq 4$.

Haynes et al. [2] studied the global defensive k-alliances in paths and obtained some results for the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number*. Thus, for any path P_n with $n \geq 3$ they proved that the *global defensive* 0-*alliance number* is equal to the total domination number, *i.e.* $\gamma_0^d(P_n) = \gamma_t(P_n)$. Furthermore, for the *global defensive* (-1)-*alliance number*, they showed that: for $n \geq 2$, $\gamma_{-1}^d(P_n) = \gamma_t(P_n)$ unless $n \equiv 2 (\text{mod } 4)$, in which case $\gamma_{-1}^d(P_n) = \gamma_t(P_n) - 1$.

2.1.9. Line graphs

A line graph $\mathcal{L}(G)$ of a graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges in G meet at one or both endpoints. In this part, we present theoretical results concerning defensive k-alliance parameters in line graphs. Let $G = (V, E)$ be a graph of size m and degree sequence $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Let $\mathcal{L}(G)$ be the line graph of G.

Sigarreta and Rodríguez-Velázquez [36] studied mathematical properties of the *defensive* (-1)-*alliance number*, the *defensive* 0-*alliance number*, the *global defensive* (-1)-*alliance number* and the *global defensive* 0-*alliance number* in line graphs. They obtained bounds for $a^d_{-1}(\mathcal{L}(G))$ and $a^d_0(\mathcal{L}(G))$ in terms of the maximum degree of G (δ_1), its minimum degree (δ_n) and its second minimum degree (δ_{n-1}) as follows: $\lceil \frac{\delta_n+\delta_{n-1}}{2} \rceil \leq a^d_0(\mathcal{L}(G)) \leq \delta_1$, $\left\lceil \frac{\delta_n+\delta_{n-1}-1}{2} \right\rceil \leq a^d_{-1}(\mathcal{L}(G)) \leq \delta_1$ (note that all these bounds are reached, for instance, in the case of $G = C_4$ with $a^d_{-1}(\mathcal{L}(C_4)) = a^d_0(\mathcal{L}(C_4)) = 2$). Moreover, if G has a unique vertex of maximum degree then the upper bound becomes $a^d_{-1}(\mathcal{L}(G)) \leq \delta_1 - 1$. They also showed that if G is a δ-regular graph with $\delta > 0$ then $a^d_{-1}(\mathcal{L}(G)) = a^d_0(\mathcal{L}(G)) = \delta$. Furthermore, for a simple graph G, Sigarreta and Rodríguez-Velázquez [36] gave bounds for $\gamma^d_{-1}(\mathcal{L}(G))$ and $\gamma^d_0(\mathcal{L}(G))$ by means of the maximum degrees δ_1 and δ_2 of G and its size m. These bounds are: $\gamma^d_{-1}(\mathcal{L}(G)) \geq \left\lceil \frac{2m}{\delta_1+\delta_2+1} \right\rceil$, $\gamma^d_0(\mathcal{L}(G)) \geq \left\lceil \frac{2m}{\delta_1+\delta_2} \right\rceil$, and if $m > 6$ then $\gamma^d_{-1}(\mathcal{L}(G)) \geq \lceil \sqrt{m+4} - 1 \rceil$.

Rodríguez-Velázquez et al. [34] and Sigarreta [43] studied the defensive k-alliances and obtained bounds for the *defensive k-alliance number* in $\mathcal{L}(G)$. They showed that for every $k \in \{2 - \delta_1 - \delta_2, \ldots, \delta_1 + \delta_2 - 2\}$, $a^d_k(\mathcal{L}(G)) \geq \left\lceil \frac{\delta_n+\delta_{n-1}+k}{2} \right\rceil$ (note that this bound is attained for instance for the graph $\mathcal{L}(K_4)$ for every $k \in \{2 - \delta_1 - \delta_2, \ldots, 0\}$, see Fig. 1(h) of Appendix in which $a^d_0(\mathcal{L}(K_4)) = 3$). Moreover, they proved that for every $k \in \{2(1 - \delta_1), \ldots, 0\}$, $a^d_k(\mathcal{L}(G)) \leq \delta_1 + \lceil \frac{k}{2} \rceil$; note that this upper bound is attained if G is a δ-regular graph [34,43]. Furthermore, Sigarreta [43] established an other lower bound by involving the algebraic connectivity μ_l of line graph $\mathcal{L}(G)$. He proved that the *defensive k-alliance number* is bounded by $a^d_k(\mathcal{L}(G)) \geq \left\lceil \frac{m(\mu_l - \lfloor \frac{\delta_1+\delta_2-2-k}{2} \rfloor)}{\mu_l} \right\rceil$.

Fernau et al. [44], Rodríguez-Velázquez and Sigarreta [37] and Sigarreta in his thesis [43] presented a lower bound for the *global defensive k-alliance number* in $\mathcal{L}(G)$ by using the maximum degrees δ_1 and δ_2 of G and its size m. Thus they obtained that $\gamma^d_k(\mathcal{L}(G)) \geq \left\lceil \frac{m}{\lfloor \frac{\delta_1+\delta_2-2-k}{2} \rfloor + 1} \right\rceil$. Moreover, Sigarreta [43] established two other lower bounds for $\gamma^d_k(\mathcal{L}(G))$. In fact, he obtained that for all graph G of size m and degree sequence $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, $\gamma^d_k(\mathcal{L}(G)) \geq \left\lceil \frac{m}{\sqrt{(\delta_1+\delta_2-2)(\delta_1+\delta_3-2)} - k + 1} \right\rceil$; Furthermore, if there exist in G two non adjacent vertices whose degrees are δ_1 and δ_2 then $\gamma^d_k(\mathcal{L}(G)) \geq \left\lceil \frac{m}{\delta_1+\delta_2-k-1} \right\rceil$.

2.1.10. Cartesian product graphs

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with the sets of vertices $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \ldots, u_{n_2}\}$ respectively, the Cartesian product of G_1 and G_2 is the graph $G_1 \times G_2 = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_l) are adjacent in $G_1 \times G_2$ if and only if "$v_i = v_k$ and $(u_j, u_l) \in E_2$" or "$(v_i, v_k) \in E_1$ and $u_j = u_l$". Let G_i be a graph of order n_i, minimum degree $\bar{\delta}_i$ and maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$.

Kristiansen et al. [1] studied the defensive k-alliances in Cartesian product graphs and obtained bounds for the *defensive* (-1)-*alliance number* and the *defensive* 0-*alliance number* as follows: $a^d_{-1}(G_1 \times G_2) \leq \min\{a^d_{-1}(G_1)a^d_0(G_2), a^d_0(G_1)a^d_{-1}(G_2)\}$ and $a^d_0(G_1 \times G_2) \leq a^d_0(G_1)a^d_0(G_2)$.

Chang et al. [49] obtained a lower bound for the *global defensive* (-1)-*alliance number* of a general graph. They showed that if G is a graph of order n and maximum degree δ_1, then $\gamma^d_{-1}(G) \geq \left\lceil \frac{2n}{\lfloor \frac{\delta_1+3}{2} \rfloor} \right\rceil$. As a consequence they established a lower bound for the *global defensive* (-1)-*alliance number* of the Cartesian product of paths and cycles. Thus, if $G_i = P_{n_i}$ or C_{n_i} for $i = 1, 2$, then $\gamma^d_{-1}(G_1 \times G_2) \geq \lceil \frac{n_1 n_2}{3} \rceil$.

Yero in his thesis [25] and Yero et al. [23] studied defensive k-alliances in Cartesian product graphs and gave some results for the *defensive k-alliance number* in $G_1 \times G_2$. They showed that if S_1 is a defensive k_1-alliance in G_1 and S_2 is a defensive k_2-alliance in G_2, then $S_1 \times S_2$ is a defensive $(k_1 + k_2)$-alliance in $G_1 \times G_2$ and $a^d_{k_1+k_2}(G_1 \times G_2) \leq a^d_{k_1}(G_1)a^d_{k_2}(G_2)$; note that this bound is a general case of the results obtained by Kristiansen et al. [1]. They also obtained that $a^d_{k-s}(G_1 \times G_2) \leq \min\{a^d_k(G_1), a^d_k(G_2)\}$ where $s \in \mathbb{Z}$ such that $\max\{\bar{\Delta}_1, \bar{\Delta}_2\} \leq s \leq \bar{\Delta}_1 + \bar{\Delta}_2 + k$. On the other hand, Yero [25] obtained that if $G_1 \times G_2$ contains defensive k-alliances, then G_i contains defensive $(k - \bar{\Delta}_j)$-alliances, with $i, j \in \{1, 2\}, i \neq j$, and as a consequence $a^d_k(G_1 \times G_2) \geq \max\{a^d_{k-\bar{\Delta}_2}(G_1), a^d_{k-\bar{\Delta}_1}(G_2)\}$.

Yero [25] and Yero et al. [23] studied global defensive k-alliances in Cartesian product graphs and presented some bounds for the *global defensive k-alliance number* in $G_1 \times G_2$. Thus, they obtained that if G_1 contains a global defensive k_1-alliance, then for every integer $k_2 \in \{-\bar{\Delta}_2, \ldots, \bar{\delta}_2\}$, $G_1 \times G_2$ contains a global defensive $(k_1 + k_2)$-alliance and $\gamma^d_{k_1+k_2}(G_1 \times G_2) \leq \gamma^d_{k_1}(G_1)n_2$. And if G_2 contains a global defensive k_2-alliance, then for every integer $k_1 \in \{-\bar{\Delta}_1, \ldots, \bar{\delta}_1\}$, $G_1 \times G_2$ contains a global defensive $(k_1 + k_2)$-alliance and $\gamma^d_{k_1+k_2}(G_1 \times G_2) \leq \gamma^d_{k_2}(G_2)n_1$.

Remark 1. Let us note that:

(i) The defensive k-alliances were studied in the literature for other graph classes such as star graphs, cubic graphs and circulant graphs. For more details, the reader can refer to [2,37,41–44].

(ii) Some results for the defensive k-alliance number $a^d_k(G)$ in the case of complement graphs are given by Sigarreta et al. [52].

Now, we summarize the results presented above by giving some bounds and exact values obtained for various parameters of defensive k-alliances for different graph classes. These results are given in Tables 1 and 2.

Concluding remarks 1. As we can see from Tables 1 and 2, the defensive k-alliance numbers are studied for various graph classes. From this, we note that the most studied parameter is the global defensive (-1)-alliance number $(\gamma^d_{-1}(G))$ and the least studied one is the upper defensive k-alliance number $(A^d_k(G))$. Furthermore, the general and tree graph classes are the most studied ones and the cycle and path graph classes are the least studied ones. Moreover, some parameters are not studied for certain graph classes. For example for the planar graphs class, several defensive k-alliance numbers are not studied such as $a^d_{-1}(P)$, $a^d_k(P)$ and $A^d_k(P)$. Besides, for the regular graphs class, all the defensive k-alliance numbers with index k namely $a^d_k(R_\delta)$, $\gamma^d_k(R_\delta)$ and $A^d_k(R_\delta)$ are not studied. However, the upper defensive k-alliance numbers are not studied for several graph classes. In particular, for $A^d_k(G)$ just some results are given in the case of general and complete (bipartite) graph classes.

Remark 2. The defensive (-1)-alliances as defined in the literature take into consideration the defense of a single vertex. In order to forestall any attack on the entire alliance or any subset of the alliance, Brigham et al. [53] proposed a model that take over this situation. Thus, they introduced the so-called secure sets as a generalization of the concept of defensive (-1)-alliances. Security and secure sets are studied in the literature and for more details one can refer to [53–55] and others.

Remark 3. Rad and Rezazadeh [56] studied (strong) open alliances in graphs. According to their definition, an alliance is called open if it is defined completely in terms of open neighborhoods. They investigated the (strong) open defensive (resp. offensive) alliances by defining parameters called (strong) open defensive (resp. offensive) alliance number denoted by $(\hat{a}_t(G))$ $a_t(G)$ (resp. $(\hat{a}_{ot}(G))$ $a_{ot}(G)$). Since $a_t(G) \cong a^d_0(G)$ and $\hat{a}_{ot}(G) \cong a^o_1(G)$, Rad and Rezazadeh [56] established bounds only for $\hat{a}_t(G)$ and $a_{ot}(G)$.

2.2. *Study of defensive k-alliance partition numbers for some graph classes*

The partitioning of graphs into k-alliances is a process which partitions the set of vertices of a graph into subsets, so that each subset constitutes a k-alliance. The problem of partitioning a graph into defensive 0-alliances is introduced and studied by Gerber and Kobler [57] and Shafique and Dutton [58], and is referred to as "Satisfactory Graph Partitioning Problem (SGP)". Thereafter, Shafique [28] investigated this partitioning problem and its application to data clustering. Moreover, Seba et al. [29] studied the partitioning of graphs into defensive (-1)-alliances and its application in mobile ad hoc networks (MANETs).

Some parameters have been defined and studied in the literature for the partitioning into defensive k-alliances, these parameters are defined as follows: For any graph $G = (V, E)$, the *(global) defensive* (-1)*-alliance partition number* of G, $(\psi^{gd}_{-1}(G))$ $\psi^d_{-1}(G)$, is defined to be the maximum number of sets in a partition of V such that each set

Table 1
Previous results on defensive k-alliance numbers for various graph classes, with $k \in \{-1, 0\}$.

Graph classes	Defensive alliance numbers					
	$a_{-1}^d(G)$	$a_0^d(G)$	$\gamma_{-1}^d(G)$	$\gamma_0^d(G)$	$A_{-1}^d(G)$	$A_0^d(G)$
G	• $a_{-1}^d(G) \le \min\{n - \left\lceil \frac{\delta_n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\}$ [1] • $a_{-1}^d(G) \ge \left\lceil \frac{n\mu}{n+\mu} \right\rceil$ [40]	• $a_0^d(G) \le \min\{n - \left\lfloor \frac{\delta_n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1\}$ [1] • $a_0^d(G) \ge \left\lceil \frac{n(\mu+1)}{n+\mu} \right\rceil$ [40] • $a_0^d(G) \ge \left\lceil \frac{n(\mu - \lfloor \frac{\delta_1}{2} \rfloor)}{\mu} \right\rceil$ [40]	• $\gamma_{-1}^d(G) \ge \frac{\sqrt{4n+1}-1}{2}$ [2] • $\gamma_{-1}^d(G) \le n - \left\lceil \frac{\delta_n}{2} \right\rceil$ [2] • $\gamma_{-1}^d(G) \ge \frac{n}{\lceil \frac{r}{2} \rceil + 1}$ [2] • $\gamma_{-1}^d(G) \ge \gamma_t(G)$ [2] • $\gamma_{-1}^d(G) \ge \left\lceil \frac{n}{\lambda+2} \right\rceil$ [40] • $\gamma_{-1}^d(G) \ge \left\lceil \frac{2n}{\delta_1+3} \right\rceil$ [40]	• $\gamma_0^d(G) \ge \sqrt{n}$ [2] • $\gamma_0^d(G) \le n - \left\lfloor \frac{\delta_n}{2} \right\rfloor$ [2] • $\gamma_0^d(G) \ge \gamma_t(G)$ [2] • $\gamma_0^d(G) \ge \left\lceil \frac{n}{\lambda+1} \right\rceil$ [40] • $\gamma_0^d(G) \ge \left\lceil \frac{n}{\lfloor \frac{\delta_1}{2} \rfloor + 1} \right\rceil$ [40]		
T	• $a_{-1}^d(T) = 1$ [1]	• $a_0^d(T) \le n$ [1]	• $\gamma_{-1}^d(T) \ge \frac{n+2}{4}$ [2] • $\gamma_{-1}^d(T) \le \frac{3n}{5}$ [2] • $\lvert S \rvert \ge \lceil \frac{n+2c}{4} \rceil$ [45] • $\gamma_{-1}^d(T) \le \frac{n+s}{2}$ [46] • $\gamma_{-1}^d(T) \ge \frac{3n-l-s+4}{8}$ [47] • $\gamma_{-1}^d(T_d) = \gamma_{-1}^d(T_{2,d}) = \lceil \frac{2n}{5} \rceil$ [48] • $t^{d-1} \lfloor \frac{t-1}{2} \rfloor + t^{d-1} + t^{d-2} \le \gamma_{-1}^d(T_{t,d}) \le t^{d-1} \lfloor \frac{t-1}{2} \rfloor + t^{d-1} + t^{d-2} + t^{d-3}$ [48] • $\gamma_{-1}^d(T) \le \beta(T)$ [50] • $\gamma_{-1}^d(T) \le \frac{n+l-1}{2}$ [50]	• $\gamma_0^d(T) \ge \frac{n+2}{3}$ [2] • $\gamma_0^d(T) \le \frac{3n}{4}$ [2] • $\lvert S \rvert \ge \lceil \frac{n+2c}{3} \rceil$ [45] • $\gamma_0^d(T) \ge \frac{3n-l-s+4}{6}$ [47] • $\gamma_0^d(T) \le \frac{3\beta(T)-1}{2}$ [50] • $\gamma_0^d(T) \le \beta(T)+s-1$ [50]		
P			• $\lvert S \rvert \ge \left\lceil \frac{n+6}{6} \right\rceil$ [33,51] • $\gamma_{-1}^d(P) \ge \left\lceil \frac{n+12}{8} \right\rceil$ [45] • $\lvert S \rvert \ge \left\lceil \frac{n-2f+4}{4} \right\rceil$ [45] • $\lvert S \rvert \ge \left\lceil \frac{\sigma-7+\sqrt{(\sigma-7)^2+4(12+n)}}{2} \right\rceil$ [45]	• $\lvert S \rvert \ge \left\lceil \frac{n+6}{5} \right\rceil$ [33,51] • $\gamma_0^d(P) \ge \left\lceil \frac{n+12}{7} \right\rceil$ [45] • $\lvert S \rvert \ge \left\lceil \frac{n-2f+4}{3} \right\rceil$ [45]		
K_n	• $a_{-1}^d(K_n) = \left\lceil \frac{n}{2} \right\rceil$ [1]	• $a_0^d(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ [1]	• $\gamma_{-1}^d(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ [2]	• $\gamma_0^d(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$ [2]		
• B • $K_{r,s}$	• $a_{-1}^d(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$ [1]	• $a_0^d(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil$ [1]	• $\gamma_{-1}^d(B) \ge \frac{2n}{\delta_1+3}$ [2] • $\gamma_{-1}^d(K_{1,s}) = \left\lfloor \frac{s}{2} \right\rfloor + 1$ [2] • $\gamma_{-1}^d(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$ [2]	• $\gamma_0^d(B) \ge \frac{2n}{\delta_1+2}$ [2] • $\gamma_0^d(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil$ [2]		

(*continued on next page*)

Table 1 (*continued*)

Graph classes	Defensive alliance numbers					
	$a^d_{-1}(G)$	$a^d_0(G)$	$\gamma^d_{-1}(G)$	$\gamma^d_0(G)$	$A^d_{-1}(G)$	$A^d_0(G)$
R_δ	• $a^d_{-1}(R_\delta) = \delta$, $\delta = 1$ or 2 [1,41] • $a^d_{-1}(R_3) = 2$ [1,41] • $a^d_{-1}(R_\delta) = girth(R_\delta)$, $\delta = 4$ or 5 [1,41]	• $a^d_0(R_\delta) = 2$, $\delta = 1$ or 2 [1,41] • $a^d_0(R_\delta) = girth(R_\delta)$, $\delta = 3$ or 4 [1,41]	• $\gamma^d_{-1}(R_4) \geq \lceil \frac{n}{3} \rceil$ [2]		• $A^d_{-1}(R_\delta) = \delta$, $\delta = 1$ or 2 [41] • $A^d_{-1}(R_3) = 2$ [1,41] • $A^d_{-1}(R_\delta) = lc(R_\delta)$, $\delta = 4$ or 5 [1,41]	• $A^d_0(R_\delta) = 2$, $\delta = 1$ or 2 [41] • $A^d_0(R_\delta) = lc(R_\delta)$, $\delta = 3$ or 4 [1,41]
C_n	• $a^d_{-1}(C_n) = 2$ [1]	• $a^d_0(C_n) = 2$ [1]	• $\gamma^d_{-1}(C_n) = \gamma_t(C_n)$ [2]	• $\gamma^d_0(C_n) = \gamma_t(C_n)$ [2]	• $A^d_{-1}(C_n) = 2$ [1]	• $A^d_0(C_n) = 2$ [1]
P_n	• $a^d_{-1}(P_n) = 1$ [1]	• $a^d_0(P_n) = 2$ [1]	• $\gamma^d_{-1}(P_n) = \gamma_t(P_n)$ [2]	• $\gamma^d_0(P_n) = \gamma_t(P_n)$ [2]	• $A^d_{-1}(P_n) = 2$ [1]	• $A^d_0(P_n) = 2$ [1]
$\mathcal{L}(G)$	• $\left\lceil \frac{\delta_n+\delta_{n-1}-1}{2} \right\rceil \leq a^d_{-1}(\mathcal{L}(G)) \leq \delta_1$ [36]	• $\left\lceil \frac{\delta_n+\delta_{n-1}}{2} \right\rceil \leq a^d_0(\mathcal{L}(G)) \leq \delta_1$ [36]	• $\gamma^d_{-1}(\mathcal{L}(G)) \geq \left\lceil \frac{2m}{\delta_1+\delta_2+1} \right\rceil$ [36] • $\gamma^d_{-1}(\mathcal{L}(G)) \geq \lceil \sqrt{m+4} - 1 \rceil$ [36]	• $\gamma^d_0(\mathcal{L}(G)) \geq \left\lceil \frac{2m}{\delta_1+\delta_2} \right\rceil$ [36]		
$G_1 \times G_2$	• $a^d_{-1}(G_1 \times G_2) \leq \min\{a^d_{-1}(G_1)a^d_0(G_2), a^d_0(G_1)a^d_{-1}(G_2)\}$ [1]	• $a^d_0(G_1 \times G_2) \leq a^d_0(G_1)a^d_0(G_2)$ [1]	• $\gamma^d_{-1}(G_1 \times G_2) \geq \lceil \frac{n_1 n_2}{3} \rceil$ with $G_i = P_{n_i}$ or C_{n_i} for $i = 1, 2$, [49]			

Table 2
Previous results on defensive k-alliance numbers for various graph classes.

Graph classes	Defensive k-alliance numbers		
	$a_k^d(G)$	$\gamma_k^d(G)$	$A_k^d(G)$
G	• $\left\lceil \frac{\delta_n+k+2}{2} \right\rceil \le a_k^d(G) \le n - \left\lfloor \frac{\delta_n-k}{2} \right\rfloor$ [34,43] • $a_k^d(G) \le \left\lceil \frac{n+k+1}{2} \right\rceil$ [34,43] • $a_{k-2r}^d(G) + r \le a_k^d(G)$ [34,43] • $a_k^d(G) \ge \left\lceil \frac{n(\mu+k+1)}{n+\mu} \right\rceil$ [34,43] • $a_k^d(G) \ge \left\lceil \frac{n(\mu-\lfloor \frac{\delta_1-k}{2} \rfloor)}{\mu} \right\rceil$ [34,43] • $a_k^d(G) \ge \mathcal{I}(G) + k + 1$ [23,25] • $a_k^d(G) \ge \left\lceil \frac{\mu+2(k+1)}{2} \right\rceil$ [23,25]	• $\frac{\sqrt{4n+k^2}+k}{2} \le \gamma_k^d(G) \le n - \left\lceil \frac{\delta_n-k}{2} \right\rceil$ [37,43,44] • $\gamma_k^d(G) \ge \left\lceil \frac{n}{\lfloor \frac{\delta_1-k}{2} \rfloor + 1} \right\rceil$ [37,43,44] • $\gamma_{k-2r}^d(G) + r \le \gamma_k^d(G)$ [37,43] • $\gamma_k^d(G) \ge \left\lceil \frac{n}{\lambda-k+1} \right\rceil$ [43]	• $A_k^d(G) \le \left\lceil \frac{2n-\delta_n+k}{2} \right\rceil$ [35] • $A_k^d(G) \le r$ [35] • $A_k^d(G) \le \phi_k^d(G) + 1$ [35]
T		• $\lvert S \rvert \ge \left\lceil \frac{n+2c}{3-k} \right\rceil$ [37,43] • $\gamma_k^d(T) \ge \left\lceil \frac{n+2}{3-k} \right\rceil$ [37,43]	
P		• $\gamma_k^d(P) \ge \left\lceil \frac{n+12}{7-k} \right\rceil$ [37,43] • $\lvert S \rvert \ge \left\lceil \frac{n-2f+4}{3-k} \right\rceil$ [37,43]	
K_n	• $a_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$ [34,43]	• $\gamma_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$ [37,43]	• $A_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$ [35]
• B • $K_{r,s}$			• $A_k^d(K_{r,s}) = 1$ [35] • $A_k^d(K_{r,s}) = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{s+k}{2} \right\rceil$ [35] • $A_k^d(K_{r,s}) = r + s - \left\lfloor \frac{r-k}{2} \right\rfloor$ [35]
R_δ			
C_n			
P_n			
$\mathcal{L}(G)$	• $a_k^d(\mathcal{L}(G)) \ge \left\lceil \frac{\delta_n+\delta_{n-1}+k}{2} \right\rceil$ [34,43] • $a_k^d(\mathcal{L}(G)) \le \delta_1 + \left\lceil \frac{k}{2} \right\rceil$ [34,43] • $a_k^d(\mathcal{L}(G)) \ge \left\lceil \frac{m(\mu_l - \lfloor \frac{\delta_1+\delta_2-2-k}{2} \rfloor)}{\mu_l} \right\rceil$ [43]	• $\gamma_k^d(\mathcal{L}(G)) \ge \left\lceil \frac{m}{\lfloor \frac{\delta_1+\delta_2-2-k}{2} \rfloor + 1} \right\rceil$ [37,43,44] • $\gamma_k^d(\mathcal{L}(G)) \ge \left\lceil \frac{m}{\sqrt{(\delta_1+\delta_2-2)(\delta_1+\delta_3-2)}-k+1} \right\rceil$ [43] • $\gamma_k^d(\mathcal{L}(G)) \ge \left\lceil \frac{m}{\delta_1+\delta_2-k-1} \right\rceil$ [43]	
$G_1 \times G_2$	• $a_{k_1+k_2}^d(G_1 \times G_2) \le a_{k_1}^d(G_1) a_{k_2}^d(G_2)$ [23,25] • $a_{k-s}^d(G_1 \times G_2) \le \min\{a_k^d(G_1), a_k^d(G_2)\}$ [23,25] • $a_k^d(G_1 \times G_2) \ge \max\{a_{k-\bar{\Delta}_2}^d(G_1), a_{k-\bar{\Delta}_1}^d(G_2)\}$ [25]	• $\gamma_{k_1+k_2}^d(G_1 \times G_2) \le \gamma_{k_1}^d(G_1) n_2$ [23,25] • $\gamma_{k_1+k_2}^d(G_1 \times G_2) \le \gamma_{k_2}^d(G_2) n_1$ [23,25]	

of the partition is a (global) defensive (-1)-alliance [21,22]. The *(global) defensive k-alliance partition number* of G, $(\psi_k^{gd}(G))$ $\psi_k^d(G)$, $k \in \{-\delta_1, \ldots, \delta_n\}$ is defined to be the maximum number of sets in a partition of V such that each set of the partition is a (global) defensive k-alliance [25]. We say that G is *partitionable* into (global) defensive k-alliances if $(\psi_k^{gd}(G) \ge 2)$ $\psi_k^d(G) \ge 2$. Concerning the *defensive k-alliance partition number*, Yero et al. [23] have given examples of extreme cases as follows: $\psi_{-\delta_1}^d(G) = n$ where each set composed of one vertex is a defensive $(-\delta_1)$-alliance, and $\psi_\delta^d(G) = 1$ for the case of a connected δ-regular graph where V is the only defensive δ-alliance.

In this subsection, we study mathematical properties of the *(global) defensive (-1)-alliance partition number* and the *(global) defensive k-alliance partition number* by presenting important theoretical results obtained for these parameters. Essentially, we give bounds or exact values for defensive k-alliance partition numbers studied for some graph classes.

2.2.1. General graphs

Eroh and Gera [20] studied the basic properties of the *defensive* (-1)-*alliance partition number* by presenting general bounds by means of the minimum degree, the order and the girth of graph G. For a connected graph G of order $n \geq 3$, they obtained sharp bounds given as follows: $1 \leq \psi^d_{-1}(G) \leq \left\lfloor n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2} \right\rfloor$. Furthermore, they gave upper bounds by involving the minimum degree and the girth of G. Thus for a graph G having minimum degree δ_n, then $\psi^d_{-1}(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{\delta_n+1}{2} \right\rceil} \right\rfloor$, and if G is a graph with $girth(G) \geq 3$ and $\delta_n \geq 4$, then $\psi^d_{-1}(G) \leq \lfloor \frac{n}{girth(G)} \rfloor$.

On the other hand, Eroh and Gera [21] established an upper sharp bound for the *global defensive* (-1)-*alliance partition number* in a connected graph G having minimum degree δ_n. Thus, they proved that $\psi^{gd}_{-1}(G) \leq 1 + \lceil \frac{\delta_n}{2} \rceil$.

Yero [25] and Yero et al. [23] presented some relations for the *(global) defensive k-alliance partition number* by considering the cases where the degrees of vertices and k are even/odd. Thus, they obtained that if every vertex of G has even degree and k is odd, $k = 2l - 1$, then every (global) defensive $(2l - 1)$-alliance in G is a (global) defensive $(2l)$-alliance and vice versa. Hence, in such a case, $\psi^d_{2l-1}(G) = \psi^d_{2l}(G)$ and $\psi^{gd}_{2l-1}(G) = \psi^{gd}_{2l}(G)$. Analogously, if every vertex of G has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in G is a defensive $(2l + 1)$-alliance and vice versa. Hence, in such a case, $\psi^d_{2l}(G) = \psi^d_{2l+1}(G)$ and $\psi^{gd}_{2l}(G) = \psi^{gd}_{2l+1}(G)$. Furthermore, they established a relation between the defensive k-alliance numbers $a^d_k(G)$ and $\psi^d_k(G)$ by showing that their product is bounded by the order of graph, that is $a^d_k(G)\psi^d_k(G) \leq n$. From this relation, they deduced that the lower bounds on $a^d_k(G)$ lead to upper bounds on $\psi^d_k(G)$. For example, from the lower bound given for $a^d_k(G)$ by Rodríguez-Velázquez et al. [34], $a^d_k(G) \geq \lceil \frac{\delta_n+k+2}{2} \rceil$, they concluded that the *defensive k-alliance partition number* is bounded upperly by $\psi^d_k(G) \leq \begin{cases} \lfloor \frac{2n}{\delta_n + k + 2} \rfloor, & \delta_n + k \text{ even,} \\ \lfloor \frac{2n}{\delta_n + k + 3} \rfloor, & \delta_n + k \text{ odd.} \end{cases}$ Note that this latter bound for the even case is attained, for example, for the complete graph K_6 where $\psi^d_{-3}(G) = 3$ with the cardinality of each defensive (-3)-alliance is equal to 2. Furthermore, the corresponding bound for the odd case is reached, for instance, for the graph given in Fig. 1(i) of Appendix where $\psi^d_{-1}(G) = 2$.

Like the *defensive k-alliance partition number*, the *global defensive k-alliance partition number* is obtained from the relation between $\gamma^d_k(G)$ and $\psi^{gd}_k(G)$, and lower bounds of $\gamma^d_k(G)$. The relation between $\gamma^d_k(G)$ and $\psi^{gd}_k(G)$ given by Yero [25] and Yero et al. [23] is $\gamma^d_k(G)\psi^{gd}_k(G) \leq n$. By combining this relation and the lower bound obtained by Rodríguez-Velázquez and Sigarreta [37], $\gamma^d_k(G) \geq \left\lceil \frac{n}{\left\lfloor \frac{\delta_1-k}{2} \right\rfloor + 1} \right\rceil$, Yero [25] and Yero et al. [23] obtained that the *global defensive k-alliance partition number* is bounded upperly by $\psi^{gd}_k(G) \leq \left\lfloor \frac{\delta_1-k}{2} \right\rfloor + 1$. They established other bounds for the *global defensive k-alliance partition number*. Thus, they showed that for every graph G partitionable into global defensive k-alliances, $\psi^{gd}_k(G) \leq \left\lfloor \frac{\sqrt{k^2+4n}-k}{2} \right\rfloor$ and $\psi^{gd}_k(G) \leq \left\lfloor \frac{\delta_n-k+2}{2} \right\rfloor$. These latter bounds are attained, for instance, in the following cases given in [23,25]: $\psi^{gd}_{-1}(K_4 \times C_4) = 4$, $\psi^{gd}_0(K_3 \times C_4) = 3$, $\psi^{gd}_1(K_2 \times C_4) = 2$ and $\psi^{gd}_1(Ptr) = 2$, where Ptr denotes the Petersen graph. They also proved that for every $k \in \{1 - \delta_n, \ldots, \delta_n\}$ if $\psi^{gd}_k(G) \geq 2$, then $\gamma^d_k(G) + \psi^{gd}_k(G) \leq \frac{n+4}{2}$. By involving the algebraic connectivity μ, Yero [25] and Yero et al. [23] showed that if any graph G is partitionable into global defensive k-alliances, then $\psi^{gd}_k(G) \leq \left\lfloor \delta_1 + 1 - \frac{\mu}{2} - k \right\rfloor$. The authors in [23,25] gave an example of equality for this latter bound when the graph $G = C_3 \times C_3$ for $k = 0$, in this case $\mu = 3$. They obtained an other bound for the same invariant by using an other parameter of the graph G which is the isoperimetric number $\mathcal{I}(G) = \min_{S \subset V : |S| \leq \frac{n}{2}} \left\{ \frac{\sum_{v \in S} \deg_{\bar{S}}(v)}{|S|} \right\}$. Thus, they proved that for any graph G, if G is partitionable into global defensive k-alliances, then $\psi^{gd}_k(G) \leq \delta_1 + 1 - \mathcal{I}(G) - k$.

2.2.2. Tree graphs

Eroh and Gera [20] obtained upper and lower sharp bounds for the *defensive* (-1)-*alliance partition number* in trees. Thus, they showed that for a tree T of order $n \geq 3$, $\psi^d_{-1}(T) \leq \lfloor \frac{3n}{4} + \frac{1}{2} \rfloor$. Moreover, if T is a tree of order $n \geq 3$ and diameter $\mathcal{D} \geq 2$ then $\psi^d_{-1}(T) \geq \lceil \frac{\mathcal{D}}{2} \rceil + 1$. Furthermore, they proved that if T is a binary tree with a maximum matching M (a matching is a subset $M \subset E$ such that: $u \cap v = \varnothing$ for each $u, v \in M$), then $\psi^d_{-1}(T) \geq n - |M|$.

On the other hand, Eroh and Gera [21] showed that in a tree T of order $n \geq 3$, the *global defensive (-1)-alliance partition number* is bounded by $1 \leq \psi^{gd}_{-1}(T) \leq 2$.

2.2.3. Regular graphs

Eroh and Gera [20] studied the *defensive (-1)-alliance partition number* in regular graphs and obtained some upper bounds and an exact value for this parameter. Thus, for a δ-regular graph R_δ of order n, $\psi^{d}_{-1}(R_\delta) \leq \left\lfloor \frac{n}{\lceil \frac{\delta+1}{2} \rceil} \right\rfloor$, if furthermore $\delta \geq 3$ and $girth(R_\delta) \geq 5$, then $\psi^{d}_{-1}(R_\delta) \leq \frac{n}{1+(girth(R_\delta)-2)\lceil \frac{\delta-3}{2} \rceil}$. As particular case, for a connected 3-regular graph having a maximum matching M, $\psi^{d}_{-1}(R_3) = |M|$.

2.2.4. Cartesian product graphs

Haynes and Lachniet [22] studied the *defensive (-1)-alliance partition number* of grid graphs $P_r \times P_c$ and showed that if $4 \leq r \leq c$, then $\psi^{d}_{-1}(P_r \times P_c) = \lfloor \frac{r-2}{2} \rfloor \lfloor \frac{c-2}{2} \rfloor + r + c - 2$.

Yero [25] and Yero et al. [23] studied the *defensive k-alliance partition number* in Cartesian product graphs and they proved that for any graphs G_1 and G_2, if there exists a partition of G_i into defensive k_i-alliances, $i \in \{1, 2\}$, then there exists a partition of $G_1 \times G_2$ into defensive $(k_1 + k_2)$-alliances and $\psi^{d}_{k_1+k_2}(G_1 \times G_2) \geq \psi^{d}_{k_1}(G_1)\psi^{d}_{k_2}(G_2)$. Moreover, for any graphs G_i of order n_i and maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$, they also showed that if $s \in \mathbb{Z}$ such that $\max\{\bar{\Delta}_1, \bar{\Delta}_2\} \leq s \leq \bar{\Delta}_1 + \bar{\Delta}_2 + k$, then $\psi^{d}_{k-s}(G_1 \times G_2) \geq \max\{n_2\psi^{d}_{k}(G_1), n_1\psi^{d}_{k}(G_2)\}$.

Furthermore, Yero [25] and Yero et al. [23] proved that if G_i is partitioned into global defensive k_i-alliances, $i \in \{1, 2\}$, then the *global defensive k-alliance partition number* of $G_1 \times G_2$ is bounded by $\psi^{gd}_{k_1+k_2}(G_1 \times G_2) \geq \max\{\psi^{gd}_{k_1}(G_1), \psi^{gd}_{k_2}(G_2)\}$. Moreover, they presented a relation between the *global defensive $(k_1 + k_2)$-alliance number* of $G_1 \times G_2$ and the *global defensive k_i-alliance partition number* of G_i, $i \in \{1, 2\}$. Thus, they obtained that for a graph G_i of order n_i, $i \in \{1, 2\}$, if $\psi^{gd}_{k_i}(G) \geq 1$ then $\gamma^{d}_{k_1+k_2}(G_1 \times G_2) \leq \frac{n_1 n_2}{\max_{i\in\{1,2\}}\{\psi^{gd}_{k_i}(G_i)\}}$.

2.2.5. Partitioning a graph into boundary defensive k-alliances

Yero [25] supposed $G = (V, E)$ a graph and $\Pi^{d}_{r}(G) = \{S_1, S_2, \ldots, S_r\}$ a partition of V into r boundary defensive k-alliances and obtained tight bounds for r. Thus, he showed that if G can be partitioned into r boundary defensive k-alliances, then $\frac{2n}{2n-\delta_n+k} \leq r \leq \frac{2n}{\delta_n+k+2}$ (note that the complete graph K_n can be partitioned into $r = \frac{2n}{n+k+1}$ boundary defensive k-alliances [25]). He also presented other tight bounds for r by using the algebraic connectivity μ and the Laplacian spectral radius μ_*, these bounds are: $\frac{2\mu_*}{2\mu_*-\delta_n+k} \leq r \leq \frac{2\mu}{2\mu-\delta_1+k}$. An example where these bounds are reached is the complete graph $G = K_n$ as mentioned in [25]. Furthermore, he proved that for a graph $G = (V, E)$ and $\mathcal{C} \subset E$ a cut set partitioning V into two boundary defensive k-alliances S and $\bar{S}$, where $k \neq \delta_1$ and $k \neq \delta_n$, then $\left\lceil \frac{2m-kn}{2(\delta_1-k)} \right\rceil \leq |S| \leq \left\lfloor \frac{2m-kn}{2(\delta_n-k)} \right\rfloor$ and $|\mathcal{C}| = \frac{2m-kn}{4}$ (note that for a δ-regular graph $|S| = \frac{n}{2}$ and $|\mathcal{C}| = \frac{n(\delta-k)}{4}$ as given in [25]). On the other hand, Yero [25] showed that if $\{X, Y\}$ is a partition of V into two boundary defensive k-alliances in G, then without loss of generality, $\left\lceil \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} + \frac{n}{2} \right\rceil \leq |X| \leq \left\lfloor \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} + \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} \right\rceil \leq |Y| \leq \left\lfloor \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} \right\rfloor$.

Now, we summarize the results presented above by giving some bounds and exact values obtained for defensive k-alliance partition numbers for some graph classes. These results are given in Table 3.

Concluding remarks 2. As we can see from Table 3, and comparing with Tables 1 and 2, we deduce that the defensive k-alliance partition numbers are studied on much less graph classes contrary to the defensive k-alliance numbers. For the studied graph classes (general, tree, regular, and Cartesian product graphs) the most studied parameter is the defensive (-1)-alliance partition number ($\psi^{d}_{-1}(G)$) and the least studied one is the global defensive (-1)-alliance partition number ($\psi^{gd}_{-1}(G)$). Furthermore, the general graph class is the most studied one and the regular graph class is the least studied one. Moreover, for the tree and regular graphs classes all the defensive k-alliance partition numbers with index k namely $\psi^{d}_{k}(G)$ and $\psi^{gd}_{k}(G)$ are not studied.

Table 3
Previous results on defensive k-alliance partition numbers for some graph classes.

Graph classes	Defensive k-alliance partition numbers			
	$\psi^{d}_{-1}(G)$	$\psi^{gd}_{-1}(G)$	$\psi^{d}_{k}(G)$	$\psi^{gd}_{k}(G)$
G	• $1 \leq \psi^{d}_{-1}(G) \leq \left\lfloor n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2} \right\rfloor$ [20] • $\psi^{d}_{-1}(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{\delta_n+1}{2} \right\rceil} \right\rfloor$ [20] • $\psi^{d}_{-1}(G) \leq \left\lfloor \frac{n}{girth(G)} \right\rfloor$ [20]	• $\psi^{gd}_{-1}(G) \leq 1 + \left\lceil \frac{\delta_n}{2} \right\rceil$ [21]	• $a^{d}_{k}(G)\psi^{d}_{k}(G) \leq n$ [23,25] • $\psi^{d}_{k}(G) \leq \begin{cases} \lfloor \frac{2n}{\delta_n+k+2} \rfloor, & \delta_n+k \text{ even}, \\ \lfloor \frac{2n}{\delta_n+k+3} \rfloor, & \delta_n+k \text{ odd}. \end{cases}$ [23,25]	• $\gamma^{d}_{k}(G)\psi^{gd}_{k}(G) \leq n$ [23,25] • $\psi^{gd}_{k}(G) \leq \left\lfloor \frac{\delta_1-k}{2} \right\rfloor + 1$ [23,25] • $\psi^{gd}_{k}(G) \leq \left\lceil \frac{\sqrt{k^2+4n}-k}{2} \right\rceil$ [23,25] • $\psi^{gd}_{k}(G) \leq \left\lfloor \frac{\delta_n-k+2}{2} \right\rfloor$ [23,25] • $\gamma^{d}_{k}(G) + \psi^{gd}_{k}(G) \leq \frac{n+4}{2}$ [23,25] • $\psi^{gd}_{k}(G) \leq \left\lfloor \delta_1 + 1 - \frac{\mu}{2} - k \right\rfloor$ [23,25] • $\psi^{gd}_{k}(G) \leq \delta_1 + 1 - \mathcal{I}(G) - k$ [23,25]
T	• $\psi^{d}_{-1}(T) \leq \left\lfloor \frac{3n}{4} + \frac{1}{2} \right\rfloor$ [20] • $\psi^{d}_{-1}(T) \geq \left\lceil \frac{\mathcal{D}}{2} \right\rceil + 1$ [20] • $\psi^{d}_{-1}(T) \geq n - \lvert M \rvert$ [20]	• $1 \leq \psi^{gd}_{-1}(T) \leq 2$ [21]		
R_δ	• $\psi^{d}_{-1}(R_\delta) \leq \left\lfloor \frac{n}{\left\lceil \frac{\delta+1}{2} \right\rceil} \right\rfloor$ [20] • $\psi^{d}_{-1}(R_\delta) \leq \frac{n}{1+(girth(R_\delta)-2)\left\lceil \frac{\delta-3}{2} \right\rceil}$ [20] • $\psi^{d}_{-1}(R_3) = \lvert M \rvert$ [20]			
$G_1 \times G_2$	• $\psi^{d}_{-1}(P_r \times P_c) = \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor + r + c - 2$ [22]		• $\psi^{d}_{k_1+k_2}(G_1 \times G_2) \geq \psi^{d}_{k_1}(G_1)\psi^{d}_{k_2}(G_2)$ [23,25] • $\psi^{d}_{k-s}(G_1 \times G_2) \geq \max\{n_2\psi^{d}_{k}(G_1), n_1\psi^{d}_{k}(G_2)\}$ [23,25]	• $\psi^{gd}_{k_1+k_2}(G_1 \times G_2) \geq \max\{\psi^{gd}_{k_1}(G_1), \psi^{gd}_{k_2}(G_2)\}$ [23,25]

3. Offensive k-alliances in graphs

In this section, we study mathematical properties of offensive k-alliances by giving bounds and/or exact values of several parameters studied for various graph classes. An offensive k-alliance in a graph $G = (V, E)$ is a set of vertices $S \subset V$ with the property that every vertex in the boundary of S has at least k more neighbors in S than it has outside of S. The case $k = 1$ (resp. $k = 2$) corresponds to the standard *offensive alliances* (resp. *strong offensive alliances*) defined in [1,4].

Several parameters have been defined and studied in the literature for offensive k-alliances, one can see [4,39,45,59–61] and others. These parameters are defined as follows: The *offensive* 1-*alliance number* known as *offensive alliance number* $a_1^o(G)$ (resp. *offensive* 2-*alliance number* known as *strong offensive alliance number* $a_2^o(G)$) is the minimum cardinality among all (critical) offensive 1-alliances (resp. offensive 2-alliances) of G [1,4]. The *global offensive* 1-*alliance number* $\gamma_1^o(G)$ (resp. *global offensive* 2-*alliance number* $\gamma_2^o(G)$) is the minimum cardinality among all (critical) global offensive 1-alliances (resp. global offensive 2-alliances) of G [5]. The *upper offensive* 1-*alliance number* $A_1^o(G)$ (resp. *upper offensive* 2-*alliance number* $A_2^o(G)$) is the maximum cardinality among all critical offensive 1-alliances (resp. offensive 2-alliances) of G [1]. The *offensive k-alliance number* $a_k^o(G)$ is the minimum cardinality among all (critical) offensive k-alliances of G [8,9]. The *global offensive k-alliance number* $\gamma_k^o(G)$ is the minimum cardinality among all (critical) global offensive k-alliances of G [44,59].

Now, we give some basic relations and observations which bind various invariants of offensive k-alliances for any graph G.

(1) $a_1^o(G) \le a_2^o(G) \le A_2^o(G)$ [1,28];
(2) $a_1^o(G) \le A_1^o(G)$ [1,28];
(3) $a_1^o(G) \le \gamma_1^o(G)$ [5];
(4) $a_2^o(G) \le \gamma_2^o(G)$ [5];
(5) $\gamma(G) \le \gamma_1^o(G) \le \gamma_2^o(G)$ [39];
(6) $a_k^o(G) \le a_{k+1}^o(G)$ [59];
(7) $a_k^o(G) \le \gamma_k^o(G)$ [25,43,59];
(8) $\gamma(G) \le \gamma_k^o(G) \le \gamma_{k+1}^o(G)$ [25,43,59].

3.1. Study of offensive k-alliance numbers for various graph classes

Like defensive k-alliances the offensive k-alliances are studied in the literature for different graph classes. In this subsection, we present important theoretical results obtained for this type of alliance. We give bounds or exact values established for offensive k-alliance numbers studied for various graph classes.

3.1.1. General graphs

In what follows, we present some theoretical results which exhibit various bounds for offensive k-alliance numbers in the case of general graphs. Let $G = (V, E)$ be a general graph of order n and size m.

Favaron et al. [4] explored the elementary properties of the offensive k-alliance numbers and they obtained bounds for the *offensive* 1-*alliance number* and the *offensive* 2-*alliance number* in general graphs. Thus they showed that: $\frac{\delta_n+1}{2} \le a_1^o(G) \le \frac{\gamma(G)+n}{2}$ and $a_2^o(G) > \frac{\delta_n+1}{2}$; if every vertex of G has odd degree then $a_1^o(G) \le \frac{n}{2}$; if $n \ge 2$ then $a_1^o(G) \le \frac{2n}{3}$, and if $n \ge 3$ then $a_2^o(G) \le \frac{5n}{6}$. Moreover, they established that if $\delta_n \ge 2$ then this latter bound becomes $a_2^o(G) \le \frac{3n}{4}$.

Rodríguez-Velázquez and Sigarreta [5] studied the global offensive k-alliances and presented several tight bounds for the *global offensive* 1-*alliance number* and the *global offensive* 2-*alliance number* in terms of several parameters of graph G. They showed that for all connected graph G of order $n \ge 2$, the *global offensive* 1-*alliance number* is bounded upperly by: $\gamma_1^o(G) \le \left\lfloor \frac{2n}{3} \right\rfloor$, $\gamma_1^o(G) \le \left\lfloor \frac{\gamma(G)+n}{2} \right\rfloor$, $\gamma_1^o(G) \le \left\lfloor \frac{n(2\mu_*-\delta_n)}{2\mu_*} \right\rfloor$ and $\gamma_1^o(G) \le \min\{n - \beta(G), \left\lfloor \frac{n+\beta(G)}{2} \right\rfloor\}$, where $\gamma(G)$ (resp. μ_* and $\beta(G)$) denotes the domination number (resp. Laplacian spectral radius and independence number) of G. Note that, these bounds are attained, for instance, for the cocktail-party graph $G = K_6 - F \cong K_{2,2,2}$ where $n = \mu_* = 6$, $\delta_n = 4$, $\beta(G) = \gamma(G) = 2$ and $\gamma_1^o(G) = 4$ [5]. Moreover, they presented an other upper bound for $\gamma_1^o(G)$ in the case of any connected graph G by means of its order and its maximum degree δ_1, that is $\gamma_1^o(G) \leq \left\lfloor \frac{2n-\delta_1}{2} \right\rfloor$. On the other hand, Rodríguez-Velázquez and Sigarreta [5] obtained tight upper bounds for the *global offensive* 2-*alliance number* by proving that for all connected graph G of order n: $\gamma_2^o(G) \leq \left\lfloor \frac{n+\gamma_2(G)}{2} \right\rfloor$, and in addition if $\delta_n \geq 2$ then $\gamma_2^o(G) \leq n - \beta(G)$ and $\gamma_2^o(G) \leq \left\lfloor \frac{5n}{6} \right\rfloor$ ($\gamma_2(G)$ denotes the 2-domination number of G which is the minimum cardinality of a two dominating set; this latter is a dominating set where every vertex in $\bar{S}$ is adjacent to at least two vertices in S). These previous results on global offensive k-alliances are also given by Sigarreta and Rodríguez-Velázquez [60]. Note that to prove some of these results Rodríguez-Velázquez and Sigarreta [5,60] used a new technique with respect to the one used by Favaron et al. [4] in their proof.

Furthermore, Rodríguez-Velázquez and Sigarreta [5,60] obtained tight lower bounds for $\gamma_1^o(G)$ and $\gamma_2^o(G)$ in terms of the order and the size of graph G, as follows: $\gamma_1^o(G) \geq \left\lceil \frac{3n-\sqrt{9n^2-8n-16m}}{4} \right\rceil$ and $\gamma_2^o(G) \geq \left\lceil \frac{3n+1-\sqrt{9n^2-10n-16m+1}}{4} \right\rceil$. By involving the maximum degree of G, these bounds are improved by the same authors to obtain: $\gamma_1^o(G) \geq \left\lceil \frac{2m+n}{3\delta_1+1} \right\rceil$ and $\gamma_2^o(G) \geq \left\lceil \frac{2(m+n)}{3\delta_1+2} \right\rceil$ (note that these two latter bounds are reached, for instance, in the case of the 3-cube graph $G = K_2 \times K_2 \times K_2$, where $\gamma_1^o(G) = \gamma_2^o(G) = 4$ as given in [5,60]). Moreover, by using the Laplacian spectral radius μ_* and the minimum degree of G, Rodríguez-Velázquez and Sigarreta [5,40,60] presented other tight lower bounds for the same parameters: $\gamma_1^o(G) \geq \left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta_n+1}{2} \right\rceil \right\rceil$ and $\gamma_2^o(G) \geq \left\lceil \frac{n}{\mu_*} (\left\lceil \frac{\delta_n}{2} \right\rceil + 1) \right\rceil$. For these two latter bounds, if G is the Petersen graph, then $\mu_* = 5$, $\gamma_1^o(G) \geq 4$ and $\gamma_2^o(G) \geq 6$ [5,40,60].

On the other hand, Rodríguez-Velázquez and Sigarreta [40] gave other lower bounds for the same parameters. They showed that for a simple graph of order n, size m and maximum degree δ_1, the *global offensive* 1-*alliance number* (resp. *global offensive* 2-*alliance number*) of G is bounded by $\gamma_1^o(G) \geq \left\lceil \frac{(2n+\delta_1+1)-\sqrt{(2n+\delta_1+1)^2-8(2m+n)}}{4} \right\rceil$ (resp. $\gamma_2^o(G) \geq \left\lceil \frac{(2n+\delta_1+2)-\sqrt{(2n+\delta_1+2)^2-16(m+n)}}{4} \right\rceil$). Note that, this bound on $\gamma_1^o(G)$ (resp. on $\gamma_2^o(G)$) is tight in the case of the complete graph K_n and the complete bipartite graph $K_{3,6}$ (resp. the complete bipartite graph $K_{3,3}$) [40].

In [60], Sigarreta and Rodríguez-Velázquez studied the offensive k-alliances with connected subgraphs and showed that for all minimal global offensive 1-alliance (resp. 2-alliance) S of G such that $\langle \bar{S} \rangle$ is connected, $|S| \geq \left\lceil \frac{3n-2}{\delta_1+3} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{4n-2}{\delta_1+4} \right\rceil$) (note that these bounds are attained, for example, for the cycle graph $G = C_3$, with $\gamma_1^o(C_3) = \gamma_2^o(C_3) = 2$). Other upper bounds for the *global offensive* 1-*alliance number* and the *global offensive* 2-*alliance number* are given by Harutyunyan [62].

Fernau et al. [59] and Sigarreta [43] studied the *(global) offensive k-alliance number* and they showed that for any simple graph G and for all $k \in \{1, \ldots, \delta_n\}$ one has $\gamma_k^o(G) \leq \left\lfloor \frac{n(2k+1)}{2k+2} \right\rfloor$, and for any graph G and for every $k \in \{2-\delta_n, \ldots, \delta_n\}$ one has $\left\lceil \frac{\delta_n+k}{2} \right\rceil \leq a_k^o(G) \leq \gamma_k^o(G) \leq n - \left\lceil \frac{\delta_n-k+2}{2} \right\rceil$ (note that these latter bounds are attained for every k in the case of the complete graph K_n as mentioned in [43,59]). Furthermore, Fernau et al. [44,59] obtained lower and upper bounds for $\gamma_k^o(G)$ by using the k-domination number $\gamma_k(G)$ of a simple graph G and its Laplacian spectral radius μ_*, that are $\left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta_n+k}{2} \right\rceil \right\rceil \leq \gamma_k^o(G) \leq \left\lfloor \frac{\gamma_k(G)+n}{2} \right\rfloor$. On the other hand, Sigarreta [43] presented two lower bounds on $\gamma_k^o(G)$ by means of the order of graph G, its size and its maximum degree. These bounds are: $\gamma_k^o(G) \geq \left\lceil \frac{2m+kn}{3\delta_1+k} \right\rceil$ and $\gamma_k^o(G) \geq \left\lceil \frac{(2n+\delta_1+k)-\sqrt{(2n+\delta_1+k)^2-8(2m+kn)}}{4} \right\rceil$ (note that these two bounds are attained, for example, for the complete graph K_n where $\gamma_k^o(G) = 1$ and $k = 3-n$). Moreover, Chellali et al. [63] obtained different bounds for $\gamma_k^o(G)$ in terms of order, maximum degree, independence number, chromatic number and minimum degree. For instance, they proved that if G is a graph of order n with minimum degree $\delta_n \geq k \geq 1$ (resp. $\delta_n \geq k+2 \geq 4$), then $\gamma_k^o(G) \leq \frac{k+1}{k+2}n$, and this bound is best possible, (resp. $\gamma_k^o(G) \leq \frac{k}{k+1}n$). Also they showed that if G is a graph of order n, minimum degree δ_n and maximum degree δ_1, then $\gamma_k^o(G) \geq \frac{n(\delta_n+k)}{2\delta_1+\delta_n+k}$. Besides, Volkmann [64] investigated the connected global offensive k-alliance number $\gamma_{k,c}^o(G)$ which is the minimum cardinality of a connected global offensive k-alliance in the graph G. In this framework, he characterized connected graphs G with $\gamma_{k,c}^o(G) = n(G)$ ($\gamma_{k,c}^o(G) = n(G) - 1$ in the case that $\delta_n \geq k \geq 2$), and presented different tight bounds for $\gamma_{k,c}^o(G)$.

Yero and Rodríguez-Velázquez [32] studied the mathematical properties of boundary powerful k-alliances and obtained that if S is a boundary offensive k-alliance in a graph G, then $\left\lceil \frac{\delta_n+k}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n-\delta_n+k-2}{2} \right\rfloor$ (note that these bounds are attained, for instance, for the complete graph $G = K_n$ for every $k \in \{3-n, \ldots, n-1\}$).

3.1.2. Tree graphs

In this paragraph, we put on view some results concerning offensive k-alliance numbers in trees. Let $T = (V, E)$ be a tree of order n.

Favaron et al. [4] studied the offensive k-alliances and explored upper bounds for the *offensive* 1-*alliance number* and the *offensive* 2-*alliance number* in trees. Thus, they obtained that for any tree of n vertices, $a_1^o(T) \leq \lfloor \frac{n}{2} \rfloor$ and $a_2^o(T) \leq \lceil \frac{3n}{4} \rceil$. For the first bound, the equality is obtained for the path and the only other examples of equality are $K_{1,3}$ with one edge subdivided once, and $K_{1,4}$ with two edges each subdivided once [4].

Rodríguez-Velázquez and Sigarreta [45] studied the global k-alliances in planar graphs and presented some results for global offensive k-alliance numbers in trees. They obtained that if S is a global offensive 2-alliance in a tree such that the subgraph $\langle S \rangle$ has c connected components, then $|S| \geq n - c + 1$. Furthermore, they showed that if S is a global offensive 1-alliance (resp. 2-alliance) in T such that $\langle \bar{S} \rangle$ is a forest with c connected components then $|S| \geq \left\lceil \frac{3(n-c)+1}{4} \right\rceil$ (resp. $|S| \geq \lceil \frac{4n-3c+1}{5} \rceil$). Moreover, Bouzefrane and Chellali [65] showed that for a tree T of order $n \geq 3$ with l leaves and s support vertices, the *global offensive* 1-*alliance number* is bounded lowerly by $\gamma_1^o(T) \geq \frac{n-l+s+1}{3}$ (with equality if and only if T belongs to a special family of trees $\mathcal{F}$ [65]). They also proved that if $T \in \mathcal{F}$ then $\gamma_1^o(T) = \gamma(T)$. On the other hand, Favaron [39] compared the *global offensive* 1-*alliance number* and the *global offensive* 2-*alliance number* to the independent domination number i. He was interested in the existence of bounds in the forms $\gamma_1^o(T) \leq f(i(T))$ and $i(T) \leq g(\gamma_2^o(T))$ where f and g are functions. Thus, he obtained that for every tree T (resp. every tree T of order $n \geq 2$), $\gamma_1^o(T) \leq 2i(T) - 1$ (resp. $i(T) \leq \gamma_2^o(T) - 1$), and these bounds are sharp.

Harutyunyan [48] studied the global offensive k-alliances in complete t-ary trees and presented an exact value for the *global offensive* 1-*alliance number*. Thus, they showed that for the complete t-ary tree $T_{t,d}$ with depth $d \geq 1$, $\gamma_1^o(T_{t,d}) = \lfloor \frac{n}{t+1} \rfloor$.

On the other hand, Chellali [66] studied the offensive k-alliances in trees and proved that if $k \geq 2$ and T belongs to a special family of trees $\mathcal{F}_k$, then $\gamma_k^o(T) = \gamma_k(T)$, with $\gamma_k(T)$ is the k-domination number of T. Moreover, Chellali and Volkmann [67] obtained an other exact value for the *global offensive* k-*alliance number* of any tree T of the family $\mathcal{F}_k$ by involving the cardinality of $L_\sigma(T)$ which is the set of vertices having degree at most $\sigma - 1$. Thus, they showed that if $T \in \mathcal{F}_k$, then $\gamma_k^o(T) = \frac{n+|L_\sigma(T)|}{2}$. Furthermore, Sigarreta [43] presented a lower bound for the cardinality of every global offensive k-alliance. He proved that if S is a global offensive k-alliance in a tree T such that the subgraph $\langle \bar{S} \rangle$ is a forest with c connected components, then $|S| \geq \left\lceil \frac{n(k+2)-3c+1}{k+3} \right\rceil$ (note that this bound is reached, for example, for the graph given in Fig. 1(j) of Appendix, where $|S| = 2$ and $c = 4$).

3.1.3. Planar graphs

In this part, we present results concerning global offensive k-alliances in planar graphs. Let $P = (V, E)$ be a planar graph of order n.

Rodríguez-Velázquez and Sigarreta [45] studied the global offensive k-alliances in planar graphs and showed that, for a planar graph P of order $n > 2$, if S is a global offensive 1-alliance (resp. 2-alliance) in P such that the subgraph $\langle \bar{S} \rangle$ has c connected components then $|S| \geq \lceil \frac{n-2c+4}{3} \rceil$ (resp. $|S| \geq \lceil \frac{n-c+2}{2} \rceil$). Moreover, for a planar graph P of order n, they proved that if S is a global offensive 1-alliance (resp. 2-alliance) in P such that the minimum degree of $\langle \bar{S} \rangle$ is at least σ then $|S| \geq \lceil \frac{n(\sigma-1)+4}{\sigma+1} \rceil$ (resp. $|S| \geq \lceil \frac{n\sigma+4}{\sigma+2} \rceil$). Furthermore, Rodríguez-Velázquez and Sigarreta [45] obtained other lower bounds for the cardinality of S which can be a global offensive 1-alliance or 2-alliance by using the number of faces of $\langle \bar{S} \rangle$. Thus, they proved that, for a planar graph of order n, if S is a global offensive 1-alliance (resp. 2-alliance) in P such that the subgraph $\langle \bar{S} \rangle$ is connected and has f faces, then $|S| \geq \left\lceil \frac{n+2f}{3} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{n+f}{2} \right\rceil$).

Sigarreta [43] considered the global offensive k-alliances and obtained lower bounds concerning the cardinality of every global offensive k-alliance in planar graphs. Thus, he showed that if S is a global offensive k-alliance in a planar graph P of order n with $k \in \{1, 2, \ldots, \delta_1\}$ (resp. $k \in \{0, 1, \ldots, \delta_1\}$) such that the subgraph $\langle \bar{S} \rangle$ has c connected components (resp. $\langle \bar{S} \rangle$ is connected with f faces), then $|S| \geq \left\lceil \frac{nk+2(2-c)}{k+2} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{nk+2f}{k+2} \right\rceil$) (note that these bounds are attained, for example, for the graph given in Fig. 1(k) of Appendix where $|S| = 3$). Furthermore, he presented a lower bound for the *global offensive* k-*alliance number*, that is for a planar graph P of order n and size m, if P contains a global offensive k-alliance of minimum cardinality greater than two, then $\gamma_k^o(P) \geq \left\lceil \frac{2m-n(6-k)+24}{6+k} \right\rceil$.

3.1.4. Complete graphs

We exhibit in this paragraph some exact values obtained for offensive k-alliance numbers in complete graphs. Let $K_n = (V, E)$ be a complete graph of order n.

Favaron et al. [4] studied the offensive k-alliances and established exact values for the *offensive* 1-*alliance number* and the *offensive* 2-*alliance number* in complete graphs. Thus, they obtained that for $n \geq 1$, $a_1^o(K_n) = \lceil \frac{n}{2} \rceil$ and $a_2^o(K_n) = \lceil \frac{n+1}{2} \rceil$. Note that these values are examples of equality in bounds given in Section 3.1.1 obtained for general graphs by Favaron et al. [4], that are $a_1^o(G) \geq \frac{\delta_n+1}{2}$ and $a_2^o(G) > \frac{\delta_n+1}{2}$.

Fernau et al. [59] obtained an exact value for the *offensive k-alliance number* in complete graphs, that is for every $k \in \{3-n, \dots, n-1\}$, $a_k^o(K_n) = \lceil \frac{n+k-1}{2} \rceil$. It is clear, in this case, that every offensive k-alliance is global and every vertex-set of cardinality $\lceil \frac{n+k-1}{2} \rceil$ is a global offensive k-alliance. Thus, as given by Bermudo et al. [68], $\gamma_k^o(K_n) = \lceil \frac{n+k-1}{2} \rceil$.

3.1.5. Bipartite graphs and complete bipartite graphs

In this part, we present theoretical results representing bounds or exact values concerning offensive k-alliance numbers in bipartite graphs and complete bipartite graphs. Let $B = (X, Y, E)$ be a bipartite graph of order n, and $K_{r,s}$ be a complete bipartite graph.

Favaron et al. [4] studied the offensive k-alliances and presented some exact values for the *offensive* 1-*alliance number* and the *offensive* 2-*alliance number* in complete bipartite graphs. Thus, they obtained that:

- for $1 \leq r \leq s$, $a_1^o(K_{r,s}) = \lceil \frac{r+1}{2} \rceil$;
- for $2 \leq r \leq s$, $a_2^o(K_{r,s}) = \lceil \frac{r}{2} + 1 \rceil$, but $a_2^o(K_{1,s}) = \lceil \frac{s}{2} + 1 \rceil$.

Note that these values are examples of equality in bounds given in Section 3.1.1 obtained for general graphs by Favaron et al. [4], that are $a_1^o(G) \geq \frac{\delta_n+1}{2}$ and $a_2^o(G) > \frac{\delta_n+1}{2}$.

Sigarreta and Rodríguez-Velázquez [60] studied the global offensive k-alliances and established an upper bound for the *global offensive* 1-*alliance number* in bipartite graphs. Thus, they obtained that for all nontrivial bipartite graph, $\gamma_1^o(B) \leq \frac{n}{2}$. Note that this bound is an improvement of the one given by Rodríguez-Velázquez and Sigarreta [5] for general graphs ($\gamma_1^o(G) \leq \lfloor \frac{2n}{3} \rfloor$) in the case of bipartite graphs. Moreover, the same authors in [69] proved that this bound is an exact value for the *global offensive* 2-*alliance number* in the case of bipartite cubic graphs, that is $\gamma_2^o(B) = \frac{n}{2}$. On the other hand, Chellali [70] obtained other bounds for $\gamma_1^o(B)$ and $\gamma_2^o(B)$. Thus, he showed that for every bipartite graph B without isolated vertices, having l vertices of degree one (and s support vertices), $\gamma_2^o(B) \leq \frac{n+l}{2}$ (and $\gamma_1^o(B) \leq \frac{n-l+s}{2}$).

By using $L_\sigma(B)$ (the set of vertices having degree at most $\sigma - 1$), Chellali and Volkmann [67] established a bound for the *global offensive k-alliance number* in bipartite graphs. They proved that for an integer $\sigma \geq 1$, one has $\gamma_k^o(B) \leq \frac{n+|L_\sigma(B)|}{2}$. On the other hand, Bermudo et al. [68] and Yero [25] gave bounds for the same parameter in complete bipartite graphs. Thus, they showed that for a complete bipartite graph $K_{r,s}$ with $s \leq r$ and for every $k \in \{2-r, \dots, r\}$:

(i) If $k \geq s+1$, then $\gamma_k^o(K_{r,s}) = r$.
(ii) If $k \leq s$ and $\lceil \frac{r+k}{2} \rceil + \lceil \frac{s+k}{2} \rceil \geq s$, then $\gamma_k^o(K_{r,s}) = s$.
(iii) If $-s < k \leq s$ and $\lceil \frac{r+k}{2} \rceil + \lceil \frac{s+k}{2} \rceil < s$, then $\gamma_k^o(K_{r,s}) = \lceil \frac{r+k}{2} \rceil + \lceil \frac{s+k}{2} \rceil$.
(iv) If $k \leq -s$ and $\lceil \frac{r+k}{2} \rceil + \lceil \frac{s+k}{2} \rceil < s$, then $\gamma_k^o(K_{r,s}) = \min\{s, 1 + \lceil \frac{r+k}{2} \rceil\}$.

3.1.6. Regular graphs

We present in this paragraph some results obtained for offensive k-alliance numbers in regular graphs. We denote by $R_\delta = (V, E)$ the δ-regular graph of order n.

Rodríguez-Velázquez and Sigarreta [69] studied mathematical properties of the global offensive k-alliance numbers of cubic graphs and presented lower and upper bounds for the *global offensive* 1-*alliance number* in δ-regular graphs. Thus, they showed that for all δ-regular graph R_δ of order n and odd degree δ, $\frac{n(\delta+1)}{3\delta+1} \leq \gamma_1^o(R_\delta) \leq \frac{n}{2}$. In the case of regular graphs of odd degree, this upper bound is an improvement of the one given by Rodríguez-Velázquez and Sigarreta [5] for general graphs ($\gamma_1^o(G) \leq \lfloor \frac{2n}{3} \rfloor$). The same authors in [60] established an upper bound for the *global offensive* 2-*alliance number* in 3-regular connected graph, that is $\gamma_2^o(R_3) \leq \lfloor \frac{3n}{4} \rfloor$.

Bermudo et al. [68] and Yero [25] investigated the relationships between global offensive k-alliances and some characteristic sets of a graph including r-dependent sets. They obtained an exact value for the *offensive k-alliance number* in δ-regular graphs, with $\delta > 0$, by using a parameter of graphs, which is the maximum cardinality of an r-dependent set $\alpha_r(R_\delta)$ (for a graph $G = (V, E)$, a set $S \subseteq V$ is an r-dependent set in G if the maximum degree of every vertex in the subgraph $\langle S \rangle$ induced by S is at most r *i.e.* $\deg_S(v) \le r, \forall\, v \in S$). Thus, they showed that for every $k \in \{1, \dots, \delta\}$, $\gamma_k^o(R_\delta) = n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(R_\delta)$.

3.1.7. Cycle graphs

Let $C_n = (V, E)$ be a cycle graph of order n. In this part, we exhibit some results obtained for offensive k-alliance numbers in this class of graphs.

Favaron et al. [4] studied the offensive k-alliances and they obtained that the *offensive* 1*-alliance number* and the *offensive* 2*-alliance number* have the same value. Thus, they established that for $n \ge 3$, $a_1^o(C_n) = a_2^o(C_n) = \lceil \frac{n}{2} \rceil$.

The problem of finding the *global offensive k-alliance number* is NP-complete. Even so, for some graphs it is possible to obtain this number [25,68]. For instance, the family of the complete graphs, the cycle graphs and the path graphs. Thus, Bermudo et al. [68] and Yero [25] obtained exact values for the *global offensive k-alliance number* in cycle graphs for small specific values of k, in terms only of the order of graph, that is $\gamma_k^o(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{for } k = 0, \\ \lceil \frac{n}{2} \rceil & \text{for } k = 1, 2. \end{cases}$

3.1.8. Path graphs

Let $P_n = (V, E)$ be a path graph of order n. Favaron et al. [4] studied the offensive k-alliances and established exact values for the *offensive* 1*-alliance number* and the *offensive* 2*-alliance number*. Thus they obtained that for $n \ge 1$, $a_1^o(P_n) = \lfloor \frac{n}{2} \rfloor$ and $a_2^o(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.

Bermudo et al. [68] and Yero [25] studied the global offensive k-alliances and presented exact values for the *global offensive k-alliance number* in path graphs for small specific values of k. Thus, they obtained that $\gamma_k^o(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{for } k = 0, \\ \lfloor \frac{n}{2} \rfloor + k - 1 & \text{for } k = 1, 2. \end{cases}$

3.1.9. Line graphs

In this part, we exhibit some results obtained for offensive k-alliance numbers in line graphs. Let $G = (V, E)$ be a graph and $\mathcal{L}(G)$ its associated line graph.

Sigarreta [43] studied the offensive k-alliances and obtained a lower bound for the *global offensive k-alliance number* in the line graph $\mathcal{L}(G)$. Thus, from a bound obtained for general graphs (that is $\gamma_k^o(G) \ge \left\lceil \frac{2m+kn}{3\delta_1+k} \right\rceil$), he deduced that for a graph G of size m having a degree sequence $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_n$, $\gamma_k^o(\mathcal{L}(G)) \ge \left\lceil \frac{\sum_{i=1}^n \delta_i^2 + m(k-2)}{3(\delta_1+\delta_2-2)+k} \right\rceil$. Furthermore, Fernau et al. [59] and Sigarreta [43] presented a lower bound for $\gamma_k^o(\mathcal{L}(G))$ where G is a δ-regular graph. Thus, they showed that if $\mathcal{L}(G)$ is a line graph of a δ-regular graph G of order n, then $\gamma_k^o(\mathcal{L}(G)) \ge \frac{n}{4} \left\lceil \frac{2(\delta-1)+k}{2} \right\rceil$ (note that this bound is attained, for instance, for the graph given in Fig. 1(l) of Appendix, with $\gamma_1^o(\mathcal{L}(G)) = \gamma_2^o(\mathcal{L}(G)) = 3$). On the other hand, Sigarreta [43] deduced that if G is a cubic graph of order n then $\frac{3n}{4} \le \gamma_2^o(\mathcal{L}(G)) = \gamma_1^o(\mathcal{L}(G)) \le n$. Note that these latter bounds are tight. For example, Sigarreta [43] mentioned that the upper one is reached in the case of the complete graph K_4: $\gamma_1^o(\mathcal{L}(K_4)) = 4 = n$. Moreover, in the case of the complete bipartite graph $K_{3,3}$, he obtained that $\gamma_1^o(\mathcal{L}(K_{3,3})) = 5$ and for the lower bound $\frac{9}{2} \le \gamma_1^o(\mathcal{L}(K_{3,3}))$.

3.1.10. Cartesian product graphs

Let $G_i = (V_i, E_i)$ be a graph of order n_i, minimum degree $\bar{\delta}_i$ and maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$.

Yero and Rodríguez-Velázquez [61] obtained various closed formulas for the *global offensive* 1*-alliance number* of several families of Cartesian product graphs, given as follows:

- For any graphs G_1 and G_2, $\gamma_1^o(G_1 \times G_2) \ge \frac{1}{2} \max\{\gamma(G_1)\gamma_1^o(G_2), \gamma_1^o(G_1)\gamma(G_2)\}$. Moreover, if G_1 has an efficient dominating set (S is an efficient dominating set if each vertex in $\bar{S}$ is adjacent to exactly one vertex in S), then $\gamma_1^o(G_1 \times G_2) \ge \gamma(G_1)\gamma_1^o(G_2)$.

- Let P_n be a path graph of order n. For every graph G of minimum degree $\bar{\delta} \geq 1$, $\gamma_1^o(G \times P_n) \geq \left\lceil \frac{(n-1)\gamma_1^o(G)}{2} \right\rceil + \left\lceil \frac{\bar{\delta}}{2} \right\rceil$.
- Let C_n be a cycle graph of order n. For every graph G, $\gamma_1^o(G \times C_n) \geq \left\lceil \frac{n\gamma_1^o(G)}{2} \right\rceil$.
- If B_i is a connected bipartite graph of order n_i, $i \in \{1, 2\}$, then $\gamma_1^o(B_1 \times B_2) \leq \frac{n_1 n_2}{2}$.
- The *global offensive* 1-*alliance number* of bamboo graph $K_r \times P_t$ is $\gamma_1^o(K_r \times P_t) = \left\lfloor \frac{rt}{2} \right\rfloor$.
- For any complete graph K_r and any path graph P_t, $\gamma_1^o(K_r \times P_t) \geq \gamma_1^o(K_r)\gamma_1^o(P_t)$.
- If G_1 is a graph partitionable into two global offensive 1-alliances X_1 and X_2 and G_2 is a graph partitionable into two global offensive 2-alliances Y_1 and Y_2, then $\gamma_1^o(G_1 \times G_2) \leq |X_1||Y_1| + |X_2||Y_2|$.
- For any torus graph $C_r \times C_t$, $\gamma_1^o(C_r \times C_t) \geq \gamma_1^o(C_r)\gamma_1^o(C_t)$.
- The *global offensive* 1-*alliance number* of the graph $K_r \times C_t$ is $\gamma_1^o(K_r \times C_t) = \left\lceil \frac{rt}{2} \right\rceil$.
- For any path graph P_r and any cycle graph C_t, $\gamma_1^o(P_r \times C_t) \geq \gamma_1^o(P_r)\gamma_1^o(C_t)$.
- Let r and t be two positive integers. If r, t have the same parity, then $\gamma_1^o(K_r \times K_t) = \left\lceil \frac{rt}{2} \right\rceil$, and if r and t have different parity then $\left\lceil \frac{rt(r+t-1)}{2(r+t)} \right\rceil \leq \gamma_1^o(K_r \times K_t) \leq \left\lceil \frac{rt}{2} \right\rceil$. Moreover, for any complete graphs $\gamma_1^o(K_r \times K_t) \geq \gamma_1^o(K_r)\gamma_1^o(K_t)$.
- Let $P_r \times P_t$ be a grid graph.
  - ▷ If r and t are even, then $\gamma_1^o(P_r \times P_t) = \frac{rt}{2}$.
  - ▷ If r is even and t is odd, then $\gamma_1^o(P_r \times P_t) = \frac{r(t-1)}{2} + \left\lceil \frac{r}{3} \right\rceil$.
  - ▷ If r and t are odd, then $\frac{(r-1)(t-1)}{2} + \left\lceil \frac{r}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq \gamma_1^o(P_r \times P_t) \leq \frac{r(t-1)}{2} + \left\lceil \frac{r}{3} \right\rceil$.

Yero in his thesis [25] obtained bounds for the *offensive k-alliance number* in Cartesian product graphs. He showed that:

(i) If S_i is an offensive k_i-alliance in $G_i = (V_i, E_i)$, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \bar{\Delta}_1, k_1 - \bar{\Delta}_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$.
(ii) If $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$, with $S_i \subset V_i$, $i \in \{1, 2\}$, then S_1 is an offensive $(k+\bar{\delta}_2)$-alliance in G_1 and S_2 is an offensive $(k + \bar{\delta}_1)$-alliance in G_2, moreover, $k \leq \min\{\bar{\Delta}_1 - \bar{\delta}_2, \bar{\Delta}_2 - \bar{\delta}_1\}$.

As a consequence, he obtained that for every $k \leq \min\{k_1 - \bar{\Delta}_2, k_2 - \bar{\Delta}_1\}$, $a_k^o(G_1 \times G_2) \leq a_{k_1}^o(G_1)a_{k_2}^o(G_2)$. Note that there is equality for the graph $C_4 \times K_4$, that is $a_{-3}^o(C_4 \times K_4) = 2 = a_0^o(C_4)a_1^o(K_4)$ [25].

On the other hand, Yero [25] studied the *global offensive k-alliance number* in Cartesian product graphs and he showed that:

(a) If S is a global offensive k-alliance in G_1, then $S \times V_2$ is a global offensive $(k - \bar{\Delta}_2)$-alliance in $G_1 \times G_2$.
(b) If $S \times V_2$ is a global offensive k-alliance in $G_1 \times G_2$, then S is a global offensive $(k + \bar{\delta}_2)$-alliance in G_1, moreover, $k \leq \bar{\Delta}_1 - \bar{\delta}_2$.

As a consequence, he obtained that for any graph G_1 and any graph G_2 of order n_2 and maximum degree $\bar{\Delta}_2$, $\gamma_{k-\bar{\Delta}_2}^o(G_1 \times G_2) \leq n_2\gamma_k^o(G_1)$. Furthermore, he established that the result given in (a) above can be simplified in the case of G_2 is a regular graph. In fact, for $G_2 = (V_2, E_2)$ a δ-regular graph, a set S is a global offensive k-alliance in G_1 if and only if $S \times V_2$ is a global offensive $(k - \delta)$-alliance in $G_1 \times G_2$.

Remark 4. Let us note that the offensive k-alliances were studied in the literature for other graph classes such as star graphs and cubic graphs [39,61,69].

Now, we summarize the results presented above by giving some bounds and exact values obtained for various offensive k-alliance numbers for different graph classes. These results are given in Table 4:

Concluding remarks 3. As we can see from Table 4, the most studied parameter is the global offensive k-alliance number ($\gamma_k^o(G)$) and the least studied one is the offensive k-alliance number ($a_k^o(G)$). Furthermore, the general and tree graph classes are the most studied ones and the line graphs class is the least studied one. Moreover, some parameters are not studied for all or certain graph classes. For example, the upper offensive 1-alliance number $A_1^o(G)$ and the upper offensive 2-alliance number $A_2^o(G)$ are not studied for all graph classes. Besides, for the line graphs class, only

Table 4
Previous results on offensive k-alliance numbers for various graph classes.

Graph classes	Offensive k-alliance numbers					
	$a_1^o(G)$	$a_2^o(G)$	$\gamma_1^o(G)$	$\gamma_2^o(G)$	$a_k^o(G)$	$\gamma_k^o(G)$
G	• $\frac{\delta_n+1}{2} \le a_1^o(G) \le \frac{\gamma(G)+n}{2}$ [4] • $a_1^o(G) \le \frac{2n}{3}$ [4]	• $a_2^o(G) > \frac{\delta_n+1}{2}$ [4] • $a_2^o(G) \le \frac{5n}{6}$ [4] • $a_2^o(G) \le \frac{3n}{4}$ [4]	• $\gamma_1^o(G) \le \left\lfloor \frac{2n}{3} \right\rfloor$ [5,60] • $\gamma_1^o(G) \le \left\lfloor \frac{\gamma(G)+n}{2} \right\rfloor$ [5,60] • $\gamma_1^o(G) \le \left\lfloor \frac{n(2\mu_*-\delta_n)}{2\mu_*} \right\rfloor$ [5,60] • $\gamma_1^o(G) \le \min\{n - \beta(G), \left\lfloor \frac{n+\beta(G)}{2} \right\rfloor\}$ [5,60] • $\gamma_1^o(G) \le \left\lfloor \frac{2n-\delta_1}{2} \right\rfloor$ [5,60] • $\gamma_1^o(G) \ge \left\lceil \frac{3n-\sqrt{9n^2-8n-16m}}{4} \right\rceil$ [5,60] • $\gamma_1^o(G) \ge \left\lceil \frac{2m+n}{3\delta_1+1} \right\rceil$ [5,60] • $\gamma_1^o(G) \ge \left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta_n+1}{2} \right\rceil \right\rceil$ [5,40,60] • $\|S\| \ge \left\lceil \frac{3n-2}{\delta_1+3} \right\rceil$ [60]	• $\gamma_2^o(G) \le \left\lfloor \frac{n+\gamma_2(G)}{2} \right\rfloor$ [5,60] • $\gamma_2^o(G) \le n - \beta(G)$ [5,60] • $\gamma_2^o(G) \le \left\lfloor \frac{5n}{6} \right\rfloor$ [5,60] • $\gamma_2^o(G) \ge \left\lceil \frac{3n+1-\sqrt{9n^2-10n-16m+1}}{4} \right\rceil$ [5,60] • $\gamma_2^o(G) \ge \left\lceil \frac{2(m+n)}{3\delta_1+2} \right\rceil$ [5,60] • $\gamma_2^o(G) \ge \left\lceil \frac{n}{\mu_*}(\left\lceil \frac{\delta_n}{2} \right\rceil + 1) \right\rceil$ [5,40,60] • $\|S\| \ge \left\lceil \frac{4n-2}{\delta_1+4} \right\rceil$ [60]	• $\left\lceil \frac{\delta_n+k}{2} \right\rceil \le a_k^o(G) \le n - \left\lceil \frac{\delta_n-k+2}{2} \right\rceil$ [43,59]	• $\left\lceil \frac{\delta_n+k}{2} \right\rceil \le \gamma_k^o(G) \le n - \left\lceil \frac{\delta_n-k+2}{2} \right\rceil$ [43,59] • $\gamma_k^o(G) \le \left\lfloor \frac{n(2k+1)}{2k+2} \right\rfloor$ [43,59] • $\left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta_n+k}{2} \right\rceil \right\rceil \le \gamma_k^o(G) \le \left\lfloor \frac{\gamma_k(G)+n}{2} \right\rfloor$ [44,59] • $\gamma_k^o(G) \ge \left\lceil \frac{2m+kn}{3\delta_1+k} \right\rceil$ [43]
T	• $a_1^o(T) \le \left\lfloor \frac{n}{2} \right\rfloor$ [4]	• $a_2^o(T) \le \left\lceil \frac{3n}{4} \right\rceil$ [4]	• $\|S\| \ge \left\lceil \frac{3(n-c)+1}{4} \right\rceil$ [45] • $\gamma_1^o(T) \ge \frac{n-l+s+1}{3}$ [65] • $\gamma_1^o(T) = \gamma(T)$ [65] • $\gamma_1^o(T) \le 2i(T) - 1$ [39] • $\gamma_1^o(T_{t,d}) = \left\lfloor \frac{n}{t+1} \right\rfloor$ [48]	• $\|S\| \ge n - c + 1$ [45] • $\|S\| \ge \left\lceil \frac{4n-3c+1}{5} \right\rceil$ [45] • $i(T) \le \gamma_2^o(T) - 1$ [39]		• $\gamma_k^o(T) = \gamma_k(T)$ [66] • $\gamma_k^o(T) = \frac{n+\|L_\sigma(T)\|}{2}$ [67] • $\|S\| \ge \left\lceil \frac{n(k+2)-3c+1}{k+3} \right\rceil$ [43]
P			• $\|S\| \ge \left\lceil \frac{n-2c+4}{3} \right\rceil$[45] • $\|S\| \ge \left\lceil \frac{n(\sigma-1)+4}{\sigma+1} \right\rceil$ [45] • $\|S\| \ge \left\lceil \frac{n+2f}{3} \right\rceil$ [45]	• $\|S\| \ge \left\lceil \frac{n-c+2}{2} \right\rceil$ [45] • $\|S\| \ge \left\lceil \frac{n\sigma+4}{\sigma+2} \right\rceil$ [45] • $\|S\| \ge \left\lceil \frac{n+f}{2} \right\rceil$ [45]		• $\|S\| \ge \left\lceil \frac{nk+2(2-c)}{k+2} \right\rceil$ [43] • $\|S\| \ge \left\lceil \frac{nk+2f}{k+2} \right\rceil$ [43] • $\gamma_k^o(P) \ge \left\lceil \frac{2m-n(6-k)+24}{6+k} \right\rceil$ [43]
K_n	• $a_1^o(K_n) = \left\lceil \frac{n}{2} \right\rceil$ [4]	• $a_2^o(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$ [4]			• $a_k^o(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil$ [59]	• $\gamma_k^o(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil$ [59,68]
• B • $K_{r,s}$	• $a_1^o(K_{r,s}) = \left\lceil \frac{r+1}{2} \right\rceil$ [4]	• $a_2^o(K_{r,s}) = \left\lceil \frac{r}{2} + 1 \right\rceil$ [4] • $a_2^o(K_{1,s}) = \left\lceil \frac{s}{2} + 1 \right\rceil$ [4]	• $\gamma_1^o(B) \le \frac{n}{2}$ [60] • $\gamma_1^o(B) \le \frac{n-l+s}{2}$ [70]	• $\gamma_2^o(B) = \frac{n}{2}$ [69] • $\gamma_2^o(B) \le \frac{n+l}{2}$ [70]		• $\gamma_k^o(B) \le \frac{n+\|L_\sigma(B)\|}{2}$ [67] • $\gamma_k^o(K_{r,s}) = r$ [25,68] • $\gamma_k^o(K_{r,s}) = s$ [25,68] • $\gamma_k^o(K_{r,s}) = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{s+k}{2} \right\rceil$ [25,68] • $\gamma_k^o(K_{r,s}) = \min\{s, 1 + \left\lceil \frac{r+k}{2} \right\rceil\}$ [25,68]

(*continued on next page*)

Table 4 (*continued*)

Graph classes	Offensive k-alliance numbers					
	$a_1^o(G)$	$a_2^o(G)$	$\gamma_1^o(G)$	$\gamma_2^o(G)$	$a_k^o(G)$	$\gamma_k^o(G)$
R_δ			• $\frac{n(\delta+1)}{3\delta+1} \le \gamma_1^o(R_\delta) \le \frac{n}{2}$ [69]	• $\gamma_2^o(R_3) \le \lfloor \frac{3n}{4} \rfloor$ [60]		• $\gamma_k^o(R_\delta) = n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(R_\delta)$ [25,68]
C_n	• $a_1^o(C_n) = \lceil \frac{n}{2} \rceil$ [4]	• $a_2^o(C_n) = \lceil \frac{n}{2} \rceil$ [4]				• $\gamma_k^o(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & k = 0 \\ \lceil \frac{n}{2} \rceil, & k = 1, 2 \end{cases}$ [25,68]
P_n	• $a_1^o(P_n) = \lfloor \frac{n}{2} \rfloor$ [4]	• $a_2^o(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ [4]				• $\gamma_k^o(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & k = 0 \\ \lfloor \frac{n}{2} \rfloor + k - 1, & k = 1, 2 \end{cases}$ [25,68]
$\mathcal{L}(G)$						• $\gamma_k^o(\mathcal{L}(G)) \ge \left\lceil \frac{\sum_{i=1}^{n} \delta_i^2 + m(k-2)}{3(\delta_1+\delta_2-2)+k} \right\rceil$ [43] • $\gamma_k^o(\mathcal{L}(G)) \ge \frac{n}{4} \left\lceil \frac{2(\delta-1)+k}{2} \right\rceil$ [43,59]
$G_1 \times G_2$			• $\gamma_1^o(G_1 \times G_2) \ge \frac{1}{2} \max\{\gamma(G_1)\gamma_1^o(G_2), \gamma_1^o(G_1)\gamma(G_2)\}$ [61] • $\gamma_1^o(B_1 \times B_2) \le \frac{n_1 n_2}{2}$ [61] • $\gamma_1^o(K_r \times P_t) \ge \gamma_1^o(K_r)\gamma_1^o(P_t)$ [61] • $\gamma_1^o(C_r \times C_t) \ge \gamma_1^o(C_r)\gamma_1^o(C_t)$ [61] • $\gamma_1^o(P_r \times C_t) \ge \gamma_1^o(P_r)\gamma_1^o(C_t)$ [61] • $\gamma_1^o(K_r \times K_t) \ge \gamma_1^o(K_r)\gamma_1^o(K_t)$ [61]		• $a_k^o(G_1 \times G_2) \ge a_{k_1}^o(G_1) a_{k_2}^o(G_2)$ [25]	• $\gamma_{k-\bar{\Delta}_2}^o(G_1 \times G_2) \ge n_2 \gamma_k^o(G_1)$ [25]

the global offensive k-alliance number $\gamma_k^o(\mathcal{L}(G))$ is investigated. However, for the global offensive 2-alliance number $\gamma_2^o(G)$ there is no result in the case of cycle, path and Cartesian product graph classes.

3.2. Study of offensive k-alliance partition numbers

Like the partitioning of graphs into defensive k-alliances the partitioning into offensive k-alliances is also studied in the literature. There are two parameters of the partitioning of graphs into offensive k-alliances which are defined as follows: for any graph $G = (V, E)$, the *(global) offensive k-alliance partition number* of G, $(\psi_k^{go}(G))$ $\psi_k^o(G)$, $k \in \{2 - \delta_1, \ldots, \delta_1\}$, is defined to be the maximum number of sets in a partition of V such that each set is (a global offensive) an offensive k-alliance [25]. We say that G is partitionable into (global) offensive k-alliances if $(\psi_k^{go}(G) \geq 2)$ $\psi_k^o(G) \geq 2$. Note that if every vertex of G has even degree and k is odd, or every vertex of G has odd degree and k is even, then every (global) offensive k-alliance in G is an offensive (a global offensive) $(k+1)$-alliance and vice versa. Hence, in such a case, $\psi_k^o(G) = \psi_{k+1}^o(G)$ and $\psi_k^{go}(G) = \psi_{k+1}^{go}(G)$ [24,25].

In this subsection, we study the mathematical properties of the offensive k-alliance partition numbers. In particular, we exhibit bounds and/or exact values obtained for the *(global) offensive k-alliance partition number* for some graph classes.

3.2.1. General graphs

Sigarreta et al. [24] and Yero [25] studied the partitioning of graphs into (global) offensive k-alliances and obtained several results. Using the relation between the *offensive k-alliance number* and the *offensive k-alliance partition number* ($a_k^o(G)\psi_k^o(G) \leq n$), they established that lower bounds on $a_k^o(G)$ lead to upper bounds on $\psi_k^o(G)$. For example, from the lower bound given by Fernau et al. in [59], that is $a_k^o(G) \geq \left\lceil \frac{\delta_n+k}{2} \right\rceil$, they obtained that $\psi_k^o(G) \leq \begin{cases} \lfloor \frac{2n}{\delta_n+k} \rfloor, & \delta_n + k \text{ even}, \\ \lfloor \frac{2n}{\delta_n+k+1} \rfloor, & \delta_n + k \text{ odd}. \end{cases}$ Note that this bound is attained, for instance, for every δ-regular graph, $\delta \geq 1$, by taking $k = 2 - \delta$. In such a case, each vertex is an offensive $(2-\delta)$-alliance and $\psi_k^o(G) = n$ as illustrated in [24,25].

Analogously, by using the relation between the *global offensive k-alliance number* and *the global offensive k-alliance partition number*, that is $\gamma_k^o(G)\psi_k^{go}(G) \leq n$, Sigarreta et al. [24] and Yero [25] established that lower bounds on $\gamma_k^o(G)$ lead to upper bounds on $\psi_k^{go}(G)$. Thus, from the lower bound presented by Bermudo et al. in [68], that is $\gamma_k^o(G) \geq \left\lceil \frac{2m+kn}{3\delta_1+k} \right\rceil$, they obtained that the *global offensive k-alliance partition number* is bounded upperly by $\psi_k^{go}(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{2m+kn}{3\delta_1+k} \right\rceil} \right\rfloor$. This bound is attained, for instance, for the circulant graph $CR(n,2)$ for $k = -2$, and if $n = 3j$ it is also attained for $k \in \{-1, 0\}$ as mentioned in [24,25]. On the other hand, they showed that if a graph G is partitionable into global offensive k-alliances, then

(i) $\psi_k^{go}(G) \leq \left\lfloor \frac{2m-n(k-4)}{2n} \right\rfloor$.
(ii) $\psi_k^{go}(G) \leq \left\lfloor \frac{\delta_n-k+4}{2} \right\rfloor$.
(iii) $\psi_k^{go}(G) \leq \left\lfloor \frac{4-k+\sqrt{k^2+2(\delta_n-k)}}{2} \right\rfloor$.

Sigarreta et al. [24] and Yero [25] also obtained that for any graph G of order n and size m, $\psi_k^{go}(G) \leq \left\lfloor \frac{6m+nk}{2m+nk} \right\rfloor$, and they noted that this bound is attained for the circulant graph $CR(5n, 2)$, where $\psi_{-2}^{go}(G) = 5$. Moreover, they established bounds for the cardinality of sets belonging to a partition. Thus, they showed that if S belongs to a partition of G into global offensive k-alliances, then $\left\lceil \frac{n(2\delta_n-\delta_1+k)}{\delta_1+2\delta_n+k} \right\rceil \leq |S| \leq \left\lfloor \frac{2n\delta_1}{\delta_1+2\delta_n+k} \right\rfloor$. These bounds are reached for the circulant graph $CR(n,2)$ which contains a partition into two global offensive 0-alliances S and $\bar{S}$, such that $|S| = \lceil \frac{n}{3} \rceil$ and $|\bar{S}| = \lfloor \frac{2n}{3} \rfloor$ [24,25]. Furthermore, they proved that for a graph G with Laplacian spectral radius μ_*, if S belongs to a partition of G into global offensive k-alliances with $-\delta_n \leq k \leq \mu_* - \delta_n$, then $\left\lceil \frac{n}{2} - \sqrt{\frac{n^2(\mu_*-k)-2nm}{4\mu_*}} \right\rceil \leq |S| \leq \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2(\mu_*-k)-2nm}{4\mu_*}} \right\rfloor$. Note that these bounds are attained for the complete graph

K_n with n is even and $k = 1$. In this case K_n is partitioned into two global offensive 1-alliances of cardinality $\frac{n}{2}$ as discussed in [24,25].

3.2.2. Cartesian product graphs

Let $G_i = (V_i, E_i)$ be a graph of order n_i, $i \in \{1, 2\}$. Sigarreta et al. [24] and Yero [25] studied the partitioning of Cartesian product graphs into offensive k-alliances and obtained that if $G_i = (V_i, E_i)$ is a graph of minimum degree $\bar{\delta}_i$ and maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$ and S_i is an offensive k_i-alliance in G_i, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \bar{\Delta}_1, k_1 - \bar{\Delta}_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$. Thus, they deduced that a partition $\Pi^o_{r_i}(G_i) = \{S_1^{(i)}, S_2^{(i)}, \dots, S_{r_i}^{(i)}\}$ of G_i into r_i offensive k_i-alliances, $i \in \{1, 2\}$, induces a partition of $G_1 \times G_2$ into $r_1 r_2$ offensive k-alliances, with $k = \min\{k_2 - \bar{\Delta}_1, k_1 - \bar{\Delta}_2\}$. This partition is formally illustrated by the following matrix given in [24,25]:

$$\Pi^o_{r_1 r_2}(G_1 \times G_2) = \left\{ \begin{matrix} S_1^{(1)} \times S_1^{(2)} & \cdots & S_1^{(1)} \times S_{r_2}^{(2)} \\ S_2^{(1)} \times S_1^{(2)} & \cdots & S_2^{(1)} \times S_{r_2}^{(2)} \\ \vdots & \ddots & \vdots \\ S_{r_1}^{(1)} \times S_1^{(2)} & \cdots & S_{r_1}^{(1)} \times S_{r_2}^{(2)} \end{matrix} \right\}.$$

As a consequence, Sigarreta et al. [24] and Yero [25] established that for any graph G_i of maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$, and for every $k \le \min\{k_1 - \bar{\Delta}_2, k_2 - \bar{\Delta}_1\}$, $\psi^o_k(G_1 \times G_2) \ge \psi^o_{k_1}(G_1)\psi^o_{k_2}(G_2)$, and they noted that for the particular case of the graph $C_4 \times K_4$, $\psi^o_{-3}(C_4 \times K_4) = 8 = 4 \cdot 2 = \psi^o_0(C_4)\psi^o_1(K_4)$.

For the *global offensive k-alliance partition number*, Sigarreta et al. [24] and Yero [25] showed that for $G_i = (V_i, E_i)$ a graph of order n_i and $\Pi^{go}_{r_i}(G_i)$ a partition of G_i into r_i global offensive k_i-alliances, $i \in \{1, 2\}$, if $x_i = \min_{S \in \Pi^{go}_{r_i}(G_i)}\{|S|\}$ and $k \le \min\{k_1, k_2\}$, then

(i) $\gamma^o_k(G_1 \times G_2) \le \min\{n_2 x_1, n_1 x_2\}$;
(ii) $\psi^{go}_k(G_1 \times G_2) \ge \max\{\psi^{go}_{k_1}(G_1), \psi^{go}_{k_2}(G_2)\}$.

They mentioned that if G_i is partitionable into global offensive k_i-alliances, for $k_i \ge 1$ and $i \in \{1, 2\}$, the bound concerning the *global offensive k-alliance partition number* is attained for $1 \le k \le \min\{k_1, k_2\}$, where $\psi^{go}_k(G_1 \times G_2) = 2 = \max\{2, 2\} = \max\{\psi^{go}_{k_1}(G_1), \psi^{go}_{k_2}(G_2)\}$. Moreover, Sigarreta et al. [24] and Yero [25] deduced that if a graph G_i of order n_i is partitionable into global offensive k_i-alliances, $i \in \{1, 2\}$, then for $k \le \min\{k_1, k_2\}$, $\gamma^o_k(G_1 \times G_2) \le \frac{n_1 n_2}{\max\{\psi^{go}_{k_1}(G_1), \psi^{go}_{k_2}(G_2)\}}$. Example of equality is obtained in [24,25] for $C_4 \times K_2$, *i.e.* $\gamma^o_1(C_4 \times K_2) = \frac{4.2}{\max\{\psi^{go}_1(C_4), \psi^{go}_1(K_2)\}} = 4$.

3.2.3. Circulant graphs — $CR(n, 2)$

Let $\mathbb{Z}_n$ be the additive group of integers modulo n and let $M \subset \mathbb{Z}_n$ such that, $i \in M$ if and only if $-i \in M$. A graph $G = (V, E)$ can be constructed as follows, the vertices of V are the elements of $\mathbb{Z}_n$ and (i, j) is an edge in E if and only if $j - i \in M$. This graph is called a *circulant of order n* and it is denoted by $CR(n, M)$. The set M is called the *set of generators* of the circulant graph. With this notation, a cycle graph is $CR(n, \{-1, 1\})$ and the complete graph is $CR(n, \mathbb{Z}_n)$.

Yero [25] in his thesis studied the partitioning of circulant graphs of type $CR(n, 2) = CR(n, \{-2, -1, 1, 2\})$ into global offensive k-alliances and obtained some theoretical results. He established two partitions of $CR(n, 2)$ as follows: if n is even, $\Pi^{go}_2(CR(n, 2)) = \{\{1, 3, 5, \dots, n-1\}, \{2, 4, 6, \dots, n\}\}$ is a partition of $CR(n, 2)$ into global offensive 0-alliances, moreover, if $n = 4j$, $\Pi^{go}_4(CR(n, 2)) = \{\{1, 5, \dots, n-3\}, \{2, 6, \dots, n-2\}, \{3, 7, \dots, n-1\}, \{4, 8, \dots, n\}\}$ is a partition of $CR(n, 2)$ into global offensive (-2)-alliances.

Furthermore, Yero [25] considered the circulant graph $G = CR(n, 2)$ and proved that:

(i) Any dominating set in G is a global offensive (-2)-alliance.
(ii) G is not partitionable into global offensive 3-alliances or global offensive 4-alliances.
(iii) $\psi^{go}_1(G) = \psi^{go}_2(G) = 2$ if and only if $n = 4j$.
(iv) $\psi^{go}_{-1}(G) = \psi^{go}_0(G) = 3$ if and only if $n = 3j$.
(v) $\psi^{go}_{-2}(G) = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$.

3.2.4. Relations between $\psi_k^{go}(G)$ and k

Sigarreta et al. [24] and Yero [25] studied relations between $\psi_k^{go}(G)$ and k and they first obtained that, for any graph G without isolated vertices, there exists $k \in \{0, \ldots, \delta_n\}$ such that G is partitionable into global offensive k-alliances. As a consequence, they deduced that any graph without isolated vertices is partitionable into global offensive 0-alliances. Furthermore, they showed that if a graph is partitionable into $r \geq 3$ global offensive k-alliances, then $k \leq 3 - r$. From this latter result, they obtained an interesting consequence which state that if G is partitionable into global offensive k-alliances for $k \geq 1$, then $\psi_k^{go}(G) = 2$. Moreover, for a graph G without isolated vertices, they deduced that if $k \in \{2 - \delta_1, \ldots, 0\}$ then the *global offensive k-alliance partition number* is bounded by $2 \leq \psi_k^{go}(G) \leq 3 - k$. Note that, for the complete graph K_n, $\psi_0^{go}(K_n) = 2$ and for the cycle graph C_{3n}, $n \geq 1$, $\psi_0^{go}(C_{3n}) = 3$ [24,25]. On the other hand, Yero [25] proved that if G is a graph of order n such that $\psi_k^{go}(G) > 2$, then for every $l \in \{1, \ldots, \psi_k^{go}(G) - 2\}$, there exists a subgraph, G_l, of G of order $n(G_l) \leq n - l\gamma_k^o(G)$ such that $\psi_{l+k}^{go}(G_l) + l \geq \psi_k^{go}(G)$.

3.2.5. Partition number and chromatic number

Motivated by the lower and upper bounds given for $\psi_k^{go}(G)$ in Section 3.2.4, that is if $k \in \{2 - \delta_1, \ldots, 0\}$ then $2 \leq \psi_k^{go}(G) \leq 3 - k$, Sigarreta et al. [24] and Yero [25] studied the limit cases $\psi_0^{go}(G) = 2$ and $\psi_0^{go}(G) = 3$. Essentially, in this study they established relationships that exist between the chromatic number of G, $\chi(G)$, and $\psi_0^{go}(G)$.

Given a positive integer t, a *t-dependent set* in G is a set of vertices of G such that no vertex in the set is adjacent to more than t vertices of the set. Thus, a *0-dependent set* in G is simply an independent set of vertices in G. Sigarreta et al. [24] and Yero [25] showed that, any set belonging to a partition of a graph into $r \geq 3$ global offensive k-alliances is a $(-k)$-dependent set. Moreover, they noted that if $k = 0$ in this result, then $r = 3$ and as a consequence, every set in a partition into three global offensive 0-alliances is an independent set, and also if $\psi_0^{go}(G) = 3$ then $\chi(G) \leq 3$. A trivial example of graph where $\psi_0^{go}(G) = 3$ and $\chi(G) = 3$ is the cycle graph C_3, and a graph where $\psi_0^{go}(G) = 3$ and $\chi(G) = 2$ is the cycle graph C_6 [24,25].

Sigarreta et al. [24] and Yero [25] also obtained that if G is a non bipartite graph and $\psi_0^{go}(G) = 3$, then $\chi(G) = 3$, and they mentioned that the complete graph K_n with $n \geq 4$ is an example of graph where $\chi(G) > 3$ and $\psi_0^{go}(G) = 2$. Moreover, they deduced that for any graph G without isolated vertices and $\chi(G) > 3$, $\psi_0^{go}(G) = 2$. On the other hand, they have given another sufficient condition for the *global offensive 0-alliance partition number* to be 2, that is for any graph G without isolated vertices containing a vertex of odd degree, it is satisfied $\psi_0^{go}(G) = 2$. Sigarreta et al. [24] and Yero [25] remarked that this latter result is equivalent to saying that if $\psi_0^{go}(G) = 3$, then every vertex in G has even degree. As a consequence of this, for k odd, every partition of G into (global) offensive k-alliances is a partition of G into (global) offensive $(k + 1)$-alliances and vice versa. This latter leads to obtain that, if $\psi_0^{go}(G) = 3$ and k is odd, then $a_k^o(G) = a_{k+1}^o(G)$, $\gamma_k^o(G) = \gamma_{k+1}^o(G)$, $\psi_k^o(G) = \psi_{k+1}^o(G)$ and $\psi_k^{go}(G) = \psi_{k+1}^{go}(G)$ [24,25].

Now, we summarize the results presented above by giving some bounds obtained for offensive k-alliance partition numbers for general, circulant and Cartesian product graph classes. These results are given in Table 5.

Concluding remarks 4. As we can see from Table 5, and comparing with Table 4, we deduce that the offensive k-alliance partition numbers are studied on much less graph classes contrary to the offensive k-alliance numbers. Furthermore, we note that only the offensive partition numbers with index k namely $\psi_k^o(G)$ and $\psi_k^{go}(G)$ that are investigated. Between these two parameters, the global offensive k-alliance partition number $\psi_k^{go}(G)$ is the most studied one. Moreover, there are only three graph classes which are addressed in this case and the general class is the most studied one.

4. Powerful *k*-alliances in graphs

In this section, we study mathematical properties of powerful k-alliances by giving bounds and/or exact values of several parameters studied for various graph classes. A powerful k-alliance is a set of vertices $S \subset V$ of a graph $G = (V, E)$, which is both defensive k-alliance and offensive $(k + 2)$-alliance. The case $k = -1$ (resp. $k = 0$) corresponds to the standard *powerful alliances* (resp. *strong powerful alliances*) defined in [1,7].

Table 5
Previous results on offensive k-alliance partition numbers for some graph classes.

Graph classes	Offensive k-alliance partition numbers	
	$\psi_k^o(G)$	$\psi_k^{go}(G)$
G	• $a_k^o(G)\psi_k^o(G) \le n$ [24,25] • $\psi_k^o(G) \le \begin{cases} \lfloor \frac{2n}{\delta_n + k} \rfloor, & \delta_n + k\text{even}, \\ \lfloor \frac{2n}{\delta_n + k + 1} \rfloor, & \delta_n + k\text{odd}. \end{cases}$ [24,25]	• $\gamma_k^o(G)\psi_k^{go}(G) \le n$ [24,25] • $\psi_k^{go}(G) \le \left\lfloor \frac{n}{\lceil \frac{2m+kn}{3\delta_1+k} \rceil} \right\rfloor$ [24,25] • $\psi_k^{go}(G) \le \left\lfloor \frac{2m-n(k-4)}{2n} \right\rfloor$ [24,25] • $\psi_k^{go}(G) \le \left\lfloor \frac{\delta_n-k+4}{2} \right\rfloor$ [24,25] • $\psi_k^{go}(G) \le \left\lfloor \frac{4-k+\sqrt{k^2+2(\delta_n-k)}}{2} \right\rfloor$ [24,25] • $\psi_k^{go}(G) \le \left\lfloor \frac{6m+nk}{2m+nk} \right\rfloor$ [24,25] • $\left\lceil \frac{n(2\delta_n-\delta_1+k)}{\delta_1+2\delta_n+k} \right\rceil \le \lvert S \rvert \le \left\lfloor \frac{2n\delta_1}{\delta_1+2\delta_n+k} \right\rfloor$ [24,25] • $\left\lceil \frac{n}{2} - \sqrt{\frac{n^2(\mu_*-k)-2nm}{4\mu_*}} \right\rceil \le \lvert S \rvert \le \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2(\mu_*-k)-2nm}{4\mu_*}} \right\rfloor$ [24,25] • $\psi_k^{go}(G) = 2$ [24,25] • $2 \le \psi_k^{go}(G) \le 3 - k$ [24,25]
$G_1 \times G_2$	• $\psi_k^o(G_1 \times G_2) \ge \psi_{k_1}^o(G_1)\psi_{k_2}^o(G_2)$ [24,25]	• $\psi_k^{go}(G_1 \times G_2) \ge \max\{\psi_{k_1}^{go}(G_1), \psi_{k_2}^{go}(G_2)\}$ [24,25]
$G = CR(n, 2)$		• $\psi_1^{go}(G) = \psi_2^{go}(G) = 2$ [25] • $\psi_{-1}^{go}(G) = \psi_0^{go}(G) = 3$ [25] • $\psi_{-2}^{go}(G) = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$ [25]

Several parameters have been defined and studied in the literature for powerful k-alliances, one can see [6,7,25,40] and others. These parameters are defined as follows: The *powerful* (-1)-*alliance number* known as *powerful alliance number* $a_{-1}^p(G)$ (resp. *powerful* 0-*alliance number* known as *strong powerful alliance number* $a_0^p(G)$) is the minimum cardinality among all (critical) powerful (-1)-alliances (resp. powerful 0-alliances) of G [6,7]. The *global powerful* (-1)-*alliance number* $\gamma_{-1}^p(G)$ (resp. *global powerful* 0-*alliance number* $\gamma_0^p(G)$) is the minimum cardinality among all (critical) global powerful (-1)-alliances (resp. global powerful 0-alliances) of G [6]. The *powerful k-alliance number* $a_k^p(G)$ is the minimum cardinality among all (critical) powerful k-alliances of G [43]. The *global powerful k-alliance number* $\gamma_k^p(G)$ is the minimum cardinality among all (critical) global powerful k-alliances of G [43].

For any graph G, we have the following relations between different powerful k-alliance numbers:

(1) $a_{-1}^p(G) \ge \max\{a_{-1}^d(G), a_{-1}^o(G)\}$ [6];
(2) $\gamma_{-1}^p(G) \ge \max\{\gamma(G), a_{-1}^p(G)\}$ [6];
(3) $a_k^p(G) \ge \max\{a_k^d(G), a_{k+2}^o(G)\}$ [43];
(4) $\gamma_k^p(G) \ge \max\{\gamma(G), a_k^p(G)\}$ [43];
(5) $\gamma_k^p(G) \ge \max\{\gamma_k^d(G), \gamma_{k+2}^o(G)\}$ [43];
(6) $\gamma_{k+1}^p(G) \ge \gamma_k^p(G)$ [43].

4.1. Study of powerful k-alliance numbers for various graph classes

In this subsection, we exhibit mathematical properties of powerful k-alliances for different graph classes. Essentially, we give bounds or exact values for powerful k-alliance numbers studied for various graph classes.

4.1.1. General graphs

In this paragraph, we present theoretical results representing bounds for powerful k-alliance numbers in general graphs. Let $G = (V, E)$ be a general graph of order n and size m.

For any connected graph G of order $n \geq 2$, Brigham et al. [6] studied powerful k-alliances and proposed a sharp upper bound for the *powerful* (-1)-*alliance number*. Thus, they used the packing number $\rho(G)$ which is the maximum cardinality of a packing in G (a subset $P \subset V$ is called a *packing* in G if for every vertex $v \in V$, $|N[v] \cap P| \leq 1$ [6,71]). They obtained that the *powerful* (-1)-*alliance number* is bounded by $a^p_{-1}(G) \leq n - \rho(G)$.

Brigham et al. [6] established lower bounds for the *global powerful* (-1)-*alliance number* by using the order of G, its maximum degree δ_1, its minimum degree δ_n and its domination number $\gamma(G)$. Thus, they showed that for any graph G, its order satisfies $n \leq (\frac{\delta_1+\delta_n+2}{\delta_n+1})\gamma^p_{-1}(G)$ which leads to obtain that $\gamma^p_{-1}(G) \geq \frac{n(\delta_n+1)}{\delta_1+\delta_n+2}$. Furthermore, they proved that $\gamma^p_{-1}(G) \geq \gamma(G) + \left\lfloor \frac{\delta_n}{2} \right\rfloor$. Moreover, they obtained a sharp upper bound for the same parameter by showing that for any graph G with no isolated vertices, $\gamma^p_{-1}(G) \leq n - \left\lfloor \frac{\delta_n}{2} \right\rfloor$. On the other hand, Rodríguez-Velázquez and Sigarreta [40] gave tight lower bounds for the *global powerful* (-1)-*alliance number* and the *global powerful* 0-*alliance number* by means of the order of a simple graph G, its size m and its spectral radius λ. Thus, they showed that $\gamma^p_{-1}(G) \geq \left\lceil \frac{2m+n}{4(\lambda+1)} \right\rceil$, $\gamma^p_0(G) \geq \left\lceil \frac{m+n}{2\lambda+1} \right\rceil$, $\gamma^p_{-1}(G) \geq \left\lceil \frac{\sqrt{2m+n}}{2} \right\rceil$ and $\gamma^p_0(G) \geq \left\lceil \frac{1+\sqrt{1+8(n+m)}}{4} \right\rceil$. Note that Rodríguez-Velázquez and Sigarreta [40] have presented graphs for which these bounds are reached.

Fernau et al. [44] studied the powerful k-alliances and established lower and upper bounds for the *global powerful* k-*alliance number*. They obtained that for any graph G, $\left\lceil \frac{\sqrt{8m+4n(k+2)+(k+1)^2}+k+1}{4} \right\rceil \leq \gamma^p_k(G) \leq n - \left\lfloor \frac{\delta_n-k}{2} \right\rfloor$. These bounds are also given by Sigarreta [43] by assuming that $k \in \{1-\delta_n, \ldots, \delta_n - 2\}$ for the upper bound (note that these bounds are reached, for example, for the cycle graph $G = C_3$ for every $k \in \{-2, -1, 0\}$). Moreover, Brigham and Dutton [72] obtained a lower bound for the same parameter, that is $\gamma^p_k(G) \geq \left\lceil \frac{\delta_1+k+1}{2} \right\rceil$. By using the spectral radius λ, Sigarreta [43] obtained that for any graph G, $\gamma^p_k(G) \geq \left\lceil \frac{2m+n(k+2)}{4\lambda-2k+2} \right\rceil$.

Yero [25] and Yero and Rodríguez-Velázquez [32] studied the mathematical properties of boundary powerful k-alliances. They obtained that if S is a boundary powerful k-alliance in a graph G, then $\left\lceil \frac{\delta_n+k+2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n-\delta_n+k}{2} \right\rfloor$. Furthermore, they showed that if S is a global boundary powerful k-alliance then $\left\lceil \frac{2m+n(k+2)}{2\delta_1+2} \right\rceil \leq |S| \leq \left\lfloor \frac{2m+n(k+2)}{2\delta_n+2} \right\rfloor$ and $\left\lceil \frac{n(2\delta_n+k+2)-2m}{2\delta_n+2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(2\delta_1+k+2)-2m}{2\delta_1+2} \right\rfloor$ (note that all these bounds are attained, for instance, for the complete graph $G = K_n$ for every $k \in \{1-n, \ldots, n-3\}$). By using the number of edges of G with one endpoint in S and the other endpoint outside of S, $\mathcal{C}$, they proved that if S is a global boundary powerful k-alliance in G, with $k \neq -1$, then $|S| = \frac{2(m+n-2\mathcal{C})+nk}{2(k+1)}$ (see the graph given in Fig. 1(m) of Appendix for illustration, where $|S| = 4$).

4.1.2. Tree graphs

We present in this part some results concerning powerful k-alliance numbers in trees. Let $T = (V, E)$ be a tree of order n.

Brigham et al. [6] initiated the study of powerful k-alliances in graphs and established a sharp upper bound for the *powerful* (-1)-*alliance number* in trees. Thus, they deduced from the result given by Meir and Moon in [71] (the domination number and the packing number of a tree are equal) that for any tree T, $a^p_{-1}(T) \leq n - \gamma(T)$. They also obtained an other sharp upper bound for the same parameter, by proving that if T is a tree of order n and $T \neq P_n$, then $a^p_{-1}(T) \leq \left\lfloor \frac{n+3}{2} \right\rfloor$.

On the other hand, Rodríguez-Velázquez and Sigarreta [45] presented bounds concerning the cardinality of every global powerful (-1)-alliance or 0-alliance in trees. They showed that if S is a global powerful (-1)-alliance (resp. 0-alliance) in T and the subgraph induced by S has c connected components, then $|S| \geq \left\lceil \frac{3n+8c-2}{12} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{2n+4c-1}{5} \right\rceil$). As a consequence, they obtained tight bounds for the *global powerful* (-1)-*alliance number* and the *global powerful* 0-*alliance number*, that are $\gamma^p_{-1}(T) \geq \left\lceil \frac{n+2}{4} \right\rceil$ and $\gamma^p_0(T) \geq \left\lceil \frac{2n+7}{5} \right\rceil$. Rodríguez-Velázquez and Sigarreta [45] have given graphs for which these bounds are attained. Note that these two latter bounds are also given by Sigarreta in his thesis [43].

Sigarreta [43] established a lower bound for the cardinality of every global powerful k-alliance in trees. He showed that if S is a global powerful k-alliance in T and the subgraph $\langle S \rangle$ has c connected components, then $|S| \geq \left\lceil \frac{n(k+4)+8c-2}{2(5-k)} \right\rceil$ (note that this bound is reached, for example, for the graph given in Fig. 1(n) of Appendix, where $|S| = 5$).

4.1.3. Planar graphs

In this paragraph, we put on view bounds obtained for powerful k-alliance numbers in planar graphs. Let $P = (V, E)$ be a planar graph of order n and size m.

Rodríguez-Velázquez and Sigarreta [45] studied mathematical properties of powerful k-alliances in planar graphs and obtained tight bounds for the *global powerful* (-1)-*alliance number* and the *global powerful* 0-*alliance number* given as follows:

(i) If $n > 6$, then $\gamma^p_{-1}(P) \geq \left\lceil \frac{2m+n+48}{28} \right\rceil$.
(ii) If $n > 6$ and P is a triangle-free graph, then $\gamma^p_{-1}(P) \geq \left\lceil \frac{2m+n+32}{20} \right\rceil$.
(iii) If $n > 4$, then $\gamma^p_0(P) \geq \left\lceil \frac{m+n+24}{13} \right\rceil$.
(iv) If $n > 4$ and P is a triangle-free graph, then $\gamma^p_0(P) \geq \left\lceil \frac{m+n+16}{9} \right\rceil$.

Furthermore, they showed that if S is a global powerful (-1)-alliance (resp. 0-alliance) in a general graph G such that the subgraph $\langle S \rangle$ is planar connected with f faces, then $|S| \geq \left\lceil \frac{2m+n-8f+16}{12} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{m+n-4f+8}{5} \right\rceil$). Moreover, Rodríguez-Velázquez and Sigarreta in [45] showed that for a global powerful (-1)-alliance (resp. 0-alliance) S in a planar graph, $|S| \geq \left\lceil \frac{2m+n+48}{28} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{m+n+24}{13} \right\rceil$). Enciso and Dutton [33] and Enciso [51] proved that these bounds are increased when S is an empire. Thus, they obtained that for a planar graph P with a global powerful (-1)-alliance (resp. 0-alliance) S, if S is an empire then $|S| \geq \left\lceil \frac{2m+n+24}{20} \right\rceil$ (resp. $|S| \geq \left\lceil \frac{m+n+12}{9} \right\rceil$).

Sigarreta in his thesis [43] studied the powerful k-alliances in planar graphs and presented some bounds for the *global powerful* k-*alliance number* given as follows:

(i) If $n > 2(2-k)$, then $\gamma^p_k(P) \geq \left\lceil \frac{2(m+24)+n(k+2)}{2(13-k)} \right\rceil$.
(ii) If $n > 2(2-k)$ and P is a triangle-free graph, then $\gamma^p_k(P) \geq \left\lceil \frac{2(m+16)+n(k+2)}{2(9-k)} \right\rceil$.

Moreover, he showed that if S is a global powerful k-alliance in a general graph G such that the subgraph $\langle S \rangle$ is planar connected with f faces, then $|S| \geq \left\lceil \frac{2(m-4f+8)+n(k+2)}{2(5-k)} \right\rceil$. Note that Sigarreta [43] have given graphs for which these bounds are attained.

Yero and Rodríguez-Velázquez [32] studied the boundary powerful k-alliances and proved that if S is a global boundary powerful k-alliance in a planar connected graph with f faces, then $\left\lceil \frac{n(k+4)+2f-4}{2\delta_1+2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(k+4)+2f-4}{2\delta_n+2} \right\rfloor$. These bounds are also given by Yero in his thesis [25].

4.1.4. Complete graphs

Let $K_n = (V, E)$ be a complete graph of order n. We exhibit in this part some exact values obtained for powerful k-alliance numbers in this class of graphs.

Brigham et al. [6] studied the powerful k-alliances and they obtained that the *powerful* (-1)-*alliance number* and the *global powerful* (-1)-*alliance number* have the same exact value. Thus, they showed that for the complete graph K_n, $a^p_{-1}(K_n) = \gamma^p_{-1}(K_n) = \left\lceil \frac{n}{2} \right\rceil$. This value is also obtained by Fernau et al. [44] and Sigarreta [43] for the *global powerful* (-1)-*alliance number*.

Yero in his thesis [25] and Yero and Rodríguez-Velázquez [32] studied the boundary powerful k-alliances and proved that if S is a boundary powerful k-alliance in a complete graph K_n, then $|S| = \left\lceil \frac{n+k+1}{2} \right\rceil$.

4.1.5. Complete bipartite graphs

Let $K_{r,s} = (X, Y, E)$ be a complete bipartite graph where r (resp. s) is the cardinality of the set of vertices X (resp. Y). In this paragraph, we present some results obtained for powerful k-alliance numbers in complete bipartite graphs.

Brigham et al. [6] studied the powerful k-alliances in complete bipartite graphs and they obtained a same exact value for the *powerful* (-1)-*alliance number* and the *global powerful* (-1)-*alliance number*. Thus, they showed that for the complete bipartite graph $K_{r,s}$, $1 \leq r \leq s$, $a^p_{-1}(K_{r,s}) = \gamma^p_{-1}(K_{r,s}) = \min\{r + \left\lfloor \frac{s}{2} \right\rfloor, \left\lceil \frac{r+1}{2} \right\rceil + \left\lceil \frac{s+1}{2} \right\rceil\}$. Note that Fernau et al. [44] and Sigarreta [43] also established the same value for the *global powerful* (-1)-*alliance number*.

4.1.6. Regular graphs

We give in this paragraph some results obtained for powerful k-alliance numbers in regular graphs. We denote by $R_\delta = (V, E)$ the δ-regular graph of order n.

Brigham et al. [6] studied the powerful k-alliances in regular graphs and they obtained that if S is a powerful (-1)-alliance of a δ-regular graph R_δ, then $|\partial S| \leq |S|$. Furthermore, they established that the order of a δ-regular graph R_δ satisfies $n \leq 2\gamma_{-1}^{p}(R_\delta)$, which leads to obtain that the *global powerful* (-1)-*alliance number* is bounded by $\gamma_{-1}^{p}(R_\delta) \geq \frac{n}{2}$.

Yero [25] and Yero and Rodríguez-Velázquez [32] studied the boundary powerful k-alliances and showed that if S is a global boundary powerful k-alliance in a δ-regular graph, then $|S| = \left\lceil \frac{n(\delta+k+2)}{2(\delta+1)} \right\rceil$. They also obtained that if R_δ is a δ-regular connected planar graph with f faces and S is a global powerful k-alliance, then $|S| = \frac{n(k+4)+2f-4}{2(\delta+1)}$. Furthermore, they showed that if S is a global boundary powerful k-alliance in a δ-regular graph R_δ, with $k \neq -1$, then $|S| = \frac{n(\delta+k+2)-4\mathcal{C}}{2k+2}$ and $\mathcal{C} = \frac{n(\delta^2+2\delta-k^2-2k)}{4(\delta+1)}$, where $\mathcal{C}$ the number of edges of R_δ with one endpoint in S and the other endpoint outside of S.

4.1.7. Cycle graphs

Let $C_n = (V, E)$ be a cycle graph of order n. In this paragraph, we present some results obtained for powerful k-alliance numbers for the class of cycle graphs.

Brigham et al. [6] studied the powerful k-alliances in graphs and they obtained that the *powerful* (-1)-*alliance number* and the *global powerful* (-1)-*alliance number* have the same value. Thus, they showed that for any cycle C_n, $a_{-1}^{p}(C_n) = \gamma_{-1}^{p}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$. Moreover, Fernau et al. [44] and Sigarreta [43] established the same value written in another form for the *global powerful* (-1)-*alliance number*, that is $\gamma_{-1}^{p}(C_n) = n - \left\lfloor \frac{n}{3} \right\rfloor$.

4.1.8. Path graphs

Let $P_n = (V, E)$ be a path graph of order n. We exhibit in this part some exact values obtained for powerful k-alliance numbers in path graphs.

This class of graphs is studied by Brigham et al. [6] and they obtained that the *powerful* (-1)-*alliance number* and the *global powerful* (-1)-*alliance number* are equal in this case. Thus, they showed that for any path P_n, $a_{-1}^{p}(P_n) = \gamma_{-1}^{p}(P_n) = \left\lfloor \frac{2n}{3} \right\rfloor$. Moreover, this exact value, written in another form, is obtained by Fernau et al. [44] and Sigarreta [43] for the *global powerful* (-1)-*alliance number*, that is $\gamma_{-1}^{p}(P_n) = n - \left\lceil \frac{n}{3} \right\rceil$.

4.1.9. Cartesian product graphs

Let $G_i = (V_i, E_i)$ be a graph of order n_i, minimum degree $\bar{\delta}_i$ and maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$.

Yero [25] and Yero and Rodríguez-Velázquez [27] studied the powerful k-alliances in Cartesian product graphs and obtained some results for the associated parameters. Thus, they showed that if $S_i \subset V_i$ is a powerful k_i-alliance in G_i, $i \in \{1, 2\}$, then $S_1 \times S_2$ is a powerful k-alliance in $G_1 \times G_2$, for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \ldots, \min\{k_1 - \bar{\Delta}_2, k_2 - \bar{\Delta}_1\}\}$. As a consequence, they obtained that if G_i contains powerful k_i-alliances, $i \in \{1, 2\}$, then for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \ldots, \min\{k_1 - \bar{\Delta}_2, k_2 - \bar{\Delta}_1\}\}$, $a_k^p(G_1 \times G_2) \leq a_{k_1}^p(G_1)a_{k_2}^p(G_2)$. Furthermore, they proved that if $S_1 \subset V_1$ is a global powerful k_1-alliance in G_1, then $S_1 \times V_2$ is a global powerful k-alliance in $G_1 \times G_2$, for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \ldots, k_1 - \bar{\Delta}_2\}$. As a consequence, they obtained that if G_1 contains global powerful k_1-alliances, then for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \ldots, k_1 - \bar{\Delta}_2\}$, $\gamma_k^p(G_1 \times G_2) \leq \gamma_{k_1}^p(G_1)n_2$.

Remark 5. Let us note that the powerful k-alliances were studied for the class of cubic graphs by Sigarreta in his thesis [43]. He established some relations between $\gamma(G)$ and $\gamma_k^p(G)$, $k \in \{-3, -2, -1, 0, 1\}$, and gave lower bounds for $\gamma_{-1}^{p}(G)$ and $\gamma_0^p(G)$.

Now, we summarize the results presented above by giving some bounds and exact values obtained for powerful k-alliance numbers for different graph classes. These results are given in Table 6.

Concluding remarks 5. As we can see from Table 6, the most studied parameter is the global powerful (-1)-alliance number ($\gamma_{-1}^{p}(G)$) and the least studied one is the powerful k-alliance number ($a_k^p(G)$). Furthermore, the general and tree graph classes are the most studied ones and the regular graphs class is the least studied one. Moreover, some parameters are not studied for all or certain graph classes. For example, the powerful 0-alliance number $a_0^p(G)$, the

Table 6
Previous results on powerful k-alliance numbers for various graph classes.

Graph classes	Powerful k-alliance numbers				
	$a^p_{-1}(G)$	$\gamma^p_{-1}(G)$	$\gamma^p_0(G)$	$a^p_k(G)$	$\gamma^p_k(G)$
G	• $a^p_{-1}(G) \le n-\rho(G)$ [6]	• $\gamma^p_{-1}(G) \ge \frac{n(\delta_n+1)}{\delta_1+\delta_n+2}$ [6] • $\gamma^p_{-1}(G) \ge \gamma(G)+\left\lfloor\frac{\delta_n}{2}\right\rfloor$ [6] • $\gamma^p_{-1}(G) \le n-\left\lfloor\frac{\delta_n}{2}\right\rfloor$ [6] • $\gamma^p_{-1}(G) \ge \left\lceil\frac{2m+n}{4(\lambda+1)}\right\rceil$ [40] • $\gamma^p_{-1}(G) \ge \left\lceil\frac{\sqrt{2m+n}}{2}\right\rceil$ [40]	• $\gamma^p_0(G) \ge \left\lceil\frac{m+n}{2\lambda+1}\right\rceil$ [40] • $\gamma^p_0(G) \ge \left\lceil\frac{1+\sqrt{1+8(n+m)}}{4}\right\rceil$ [40]		• $\left\lceil\frac{\sqrt{8m+4n(k+2)+(k+1)^2}+k+1}{4}\right\rceil \le \gamma^p_k(G) \le n-\left\lceil\frac{\delta_n-k}{2}\right\rceil$ [43,44] • $\gamma^p_k(G) \ge \left\lceil\frac{\delta_1+k+1}{2}\right\rceil$ [72] • $\gamma^p_k(G) \ge \left\lceil\frac{2m+n(k+2)}{4\lambda-2k+2}\right\rceil$ [43]
T	• $a^p_{-1}(T) \le n-\gamma(T)$ [6] • $a^p_{-1}(T) \le \left\lfloor\frac{n+3}{2}\right\rfloor$ [6]	• $\|S\| \ge \left\lceil\frac{3n+8c-2}{12}\right\rceil$ [45] • $\gamma^p_{-1}(T) \ge \left\lceil\frac{n+2}{4}\right\rceil$ [43,45]	• $\|S\| \ge \left\lceil\frac{2n+4c-1}{5}\right\rceil$ [45] • $\gamma^p_0(T) \ge \left\lceil\frac{2n+7}{5}\right\rceil$ [43,45]		• $\|S\| \ge \left\lceil\frac{n(k+4)+8c-2}{2(5-k)}\right\rceil$ [43]
P		• $\gamma^p_{-1}(P) \ge \left\lceil\frac{2m+n+48}{28}\right\rceil$ [45] • $\|S\| \ge \left\lceil\frac{2m+n-8f+16}{12}\right\rceil$ [45] • $\|S\| \ge \left\lceil\frac{2m+n+24}{20}\right\rceil$ [33,51]	• $\gamma^p_0(P) \ge \left\lceil\frac{m+n+24}{13}\right\rceil$ [45] • $\|S\| \ge \left\lceil\frac{m+n-4f+8}{5}\right\rceil$ [45] • $\|S\| \ge \left\lceil\frac{m+n+12}{9}\right\rceil$ [33,51]		• $\gamma^p_k(P) \ge \left\lceil\frac{2(m+24)+n(k+2)}{2(13-k)}\right\rceil$ [43] • $\|S\| \ge \left\lceil\frac{2(m-4f+8)+n(k+2)}{2(5-k)}\right\rceil$ [43]
K_n	• $a^p_{-1}(K_n) = \left\lceil\frac{n}{2}\right\rceil$ [6]	• $\gamma^p_{-1}(K_n) = \left\lceil\frac{n}{2}\right\rceil$ [6,43,44]			
$K_{r,s}$	• $a^p_{-1}(K_{r,s}) = \min\{r+\left\lfloor\frac{s}{2}\right\rfloor, \left\lceil\frac{r+1}{2}\right\rceil+\left\lceil\frac{s+1}{2}\right\rceil\}$ [6]	• $\gamma^p_{-1}(K_{r,s}) = \min\{r+\left\lfloor\frac{s}{2}\right\rfloor, \left\lceil\frac{r+1}{2}\right\rceil+\left\lceil\frac{s+1}{2}\right\rceil\}$ [6,43,44]			
R_δ		• $\gamma^p_{-1}(R_\delta) \ge \frac{n}{2}$ [6]			
C_n	• $a^p_{-1}(C_n) = \left\lceil\frac{2n}{3}\right\rceil$ [6]	• $\gamma^p_{-1}(C_n) = \left\lceil\frac{2n}{3}\right\rceil$ [6] • $\gamma^p_{-1}(C_n) = n-\left\lfloor\frac{n}{3}\right\rfloor$ [43,44]			
P_n	• $a^p_{-1}(P_n) = \left\lfloor\frac{2n}{3}\right\rfloor$ [6]	• $\gamma^p_{-1}(P_n) = \left\lfloor\frac{2n}{3}\right\rfloor$ [6] • $\gamma^p_{-1}(P_n) = n-\left\lceil\frac{n}{3}\right\rceil$ [43,44]			
$G_1 \times G_2$				• $a^p_k(G_1\times G_2) \le a^p_{k_1}(G_1)a^p_{k_2}(G_2)$ [25,27]	• $\gamma^p_k(G_1\times G_2) \le \gamma^p_{k_1}(G_1)n_2$ [25,27]

upper powerful (-1)-alliance number $A^p_{-1}(G)$, the upper powerful 0-alliance number $A^p_0(G)$ and the upper powerful k-alliance number $A^p_k(G)$ are not studied for all graph classes (note that these three latter numbers which are not defined in the literature can be similarly defined as in the cases of defensive and offensive k-alliances). Also, the classes of bipartite graphs and line graphs are not studied for this kind of alliances. In addition, the powerful k-alliance number $a^p_k(G)$ is studied just in the case of Cartesian product graphs. Besides, in the regular (resp. Cartesian product) graphs class, just the global powerful (-1)-alliance number $\gamma^p_{-1}(G)$ (resp. $a^p_k(G)$ and $\gamma^p_k(G)$) is (resp. are) studied. On the other hand, $\gamma^p_0(G)$ and $\gamma^p_k(G)$ are not investigated for several graph classes such as the complete and complete bipartite graphs.

4.2. *Study of powerful k-alliance partition numbers*

Like partitioning of graphs into defensive k-alliances or into offensive k-alliances, the partitioning of graphs into powerful k-alliances is also studied in the literature. There are two parameters of the partitioning of graphs into powerful k-alliances which are defined as follows: For any graph $G = (V, E)$, the *(global) powerful k-alliance partition number* of G, $(\psi^{gp}_k(G))$ $\psi^p_k(G)$, $k \in \{-\delta_1, \dots, \delta_1 - 2\}$, is defined to be the maximum number of sets in a partition of V such that each set is a (global) powerful k-alliance [25,27]. We say that a graph G is partitionable into (global) powerful k-alliances if $(\psi^{gp}_k(G) \geq 2)$ $\psi^p_k(G) \geq 2$.

In this subsection, we present theoretical results obtained for the powerful k-alliance partition numbers. We give bounds and/or exact values for these parameters in general graphs and Cartesian product graphs, together with some results concerning partitions into boundary powerful k-alliances.

4.2.1. *General graphs*

Yero [25] and Yero and Rodríguez-Velázquez [27] studied the partitioning of graphs into global powerful k-alliances and established several results. They showed that, if there are two different sets in $\Pi_r(G)$ (a partition of a graph G into r dominating sets) such that one of them is a defensive k-alliance and the other one is an offensive $(k+2)$-alliance, then $k \leq 1 - r$. From this result, there are two direct and useful consequences [25,27]: the first one is that for $k \geq 0$, no graph is partitionable into global powerful k-alliances; and the second one states that if a graph G is partitionable into global powerful k-alliances, then $\psi^{gp}_k(G) \leq 1 - k$. Note that this latter bound is achieved for instance for the complete graph, which can be partitioned into two global powerful (-1)-alliances [25,27]. Furthermore, Yero [25] and Yero and Rodríguez-Velázquez [27] proved that for a graph G of order n, minimum degree δ_n and maximum degree δ_1, if G is partitionable into global powerful k-alliances then $\psi^{gp}_k(G) \leq \left\lfloor \frac{\delta_1+\delta_n+2}{\delta_n+k+2} \right\rfloor$. They noted that this bound is attained, for instance, for the complete graph $G = K_n$ where $\psi^{gp}_{-1}(G) = 2$, or for the circulant graph $G = CR(3t, 3)$ for which $\psi^{gp}_{-4}(G) = 3$. Moreover, they obtained an other upper bound in terms only of k and the order n. Thus, they showed that if G is partitionable into global powerful k-alliances, then $\psi^{gp}_k(G) \leq \left\lfloor \frac{\sqrt{8n+(2k-1)^2}-2k+1}{4} \right\rfloor$. This bound is attained, for instance, for the circulant graph $G = CR(10, 2)$ for which $\psi^{gp}_{-4}(G) = 5$ as given in [25,27], or for the graph given in Fig. 1(o) of Appendix where $\psi^{gp}_{-1}(G) = 2$.

4.2.2. *Cartesian product graphs*

Yero in his thesis [25] and Yero and Rodríguez-Velázquez [27] studied the partitioning of Cartesian product graphs into (global) powerful k-alliances. They showed that for a graph $G_i = (V_i, E_i)$ of maximum degree $\bar{\Delta}_i$, $i \in \{1, 2\}$, if G_i is partitionable into r_i powerful k_i-alliances, then the graph $G_1 \times G_2$ is partitionable into $r = r_1 r_2$ powerful k-alliances, for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \dots, \min\{k_1 - \bar{\Delta}_2, k_2 - \bar{\Delta}_1\}\}$. Furthermore, they obtained that $\psi^p_k(G_1 \times G_2) \geq \psi^p_{k_1}(G_1)\psi^p_{k_2}(G_2)$. Moreover, they established that if G_1 is partitionable into global powerful k_1-alliances, then for every $k \in \{-\bar{\Delta}_1 - \bar{\Delta}_2, \dots, k_1 - \bar{\Delta}_2\}$, $\psi^{gp}_k(G_1 \times G_2) \geq \psi^{gp}_{k_1}(G_1)$, and they remarked that if $G_1 = CR(3t, 3)$ and $G_2 = K_2$, then $\psi^{gp}_{-5}(G_1 \times G_2) = 3 = \psi^{gp}_{-4}(G_1)$.

4.2.3. *Partitioning a graph into boundary powerful k-alliances*

Yero [25] studied the partitioning of graphs into boundary powerful k-alliances and he established that every graph can be partitioned into two global boundary powerful (-1)-alliances. Thus he proved that, for a graph $G = (V, E)$:

Table 7
Previous results on powerful k-alliance partition numbers for some graph classes.

Graph classes	Powerful k-alliance partition numbers	
	$\psi_k^p(G)$	$\psi_k^{gp}(G)$
G		• $\psi_k^{gp}(G) \leq 1-k$ [25,27] • $\psi_k^{gp}(G) \leq \left\lfloor \frac{\delta_1+\delta_n+2}{\delta_n+k+2} \right\rfloor$ [25,27] • $\psi_k^{gp}(G) \leq \left\lfloor \frac{\sqrt{8n+(2k-1)^2}-2k+1}{4} \right\rfloor$ [25,27]
$G_1 \times G_2$	• $\psi_k^p(G_1 \times G_2) \geq \psi_{k_1}^p(G_1)\psi_{k_2}^p(G_2)$ [25,27]	• $\psi_k^{gp}(G_1 \times G_2) \geq \psi_{k_1}^{gp}(G_1)$ [25,27]

(i) $S \subset V$ is a global boundary powerful (-1)-alliance in G, if and only if, $\bar{S}$ is a global boundary powerful (-1)-alliance in G.
(ii) If G can be partitioned into two global boundary powerful k-alliances, then $k=-1$.

Furthermore, he obtained lower and upper bounds concerning the cardinality of every global boundary powerful (-1)-alliance in terms of the order of the graph G, its minimum degree δ_n and its maximum degree δ_1. Thus, he showed that if S is a global boundary powerful (-1)-alliance in G, then $\left\lceil \frac{n(\delta_n+1)}{\delta_1+\delta_n+2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(\delta_1+1)}{\delta_1+\delta_n+2} \right\rfloor$, and he noted that if S is a global boundary powerful (-1)-alliance in a δ-regular graph, then $|S| = \frac{n}{2}$. Moreover, Yero [25] proved that if $S \subset V$ is a global boundary powerful (-1)-alliance in a graph $G=(V,E)$ and $\mathcal{C}$ is a cut set with one endpoint in S and the other endpoint outside of S, then $\left\lceil \frac{2m+n}{2\delta_1+2} \right\rceil \leq |S| \leq \left\lfloor \frac{2m+n}{2\delta_n+2} \right\rfloor$ and $|\mathcal{C}| = \frac{2m+n}{4}$. This result leads to obtain the previous value of $|S|$ concerning the δ-regular graph. On the other hand, he obtained the result which shows the relationship between the algebraic connectivity of a graph, its Laplacian spectral radius and the respective cardinalities of the two global boundary powerful (-1)-alliances S and $\bar{S}$ which form a partition of the graph. Thus, he proved that if $S \subset V$ is a global boundary powerful (-1)-alliance in G, then without loss of generality, $\frac{n}{2} + \left\lceil \sqrt{\frac{n^2(\mu-1)-2nm}{4\mu}} \right\rceil \leq |S| \leq \frac{n}{2} + \left\lfloor \sqrt{\frac{n^2(\mu_*-1)-2nm}{4\mu_*}} \right\rfloor$ and $\frac{n}{2} - \left\lfloor \sqrt{\frac{n^2(\mu_*-1)-2nm}{4\mu_*}} \right\rfloor \leq |\bar{S}| \leq \frac{n}{2} - \left\lceil \sqrt{\frac{n^2(\mu-1)-2nm}{4\mu}} \right\rceil$, where μ (resp. μ_*) is the algebraic connectivity (resp. the Laplacian spectral radius) of the graph G. Recently, Slimani and Kheddouci [26] have introduced a new concept of saturated vertices and studied the saturated boundary k-alliances in graphs. They have proved that $S \subset V$ is a minimal global boundary powerful (-1)-alliance in G, if and only if, $\bar{S}$ is a minimal global boundary powerful (-1)-alliance in G. Furthermore, as a main result, they have obtained tight bounds for the cardinality of every minimal global boundary powerful (-1)-alliance in terms only of the order and the size of graph by taking the two cases where $\langle S \rangle$ is connected or not. Hence, they showed that for a graph $G=(V,E)$ with $|V|=n$ and $|E|=m$, if $S \subset V$ is a minimal global boundary powerful (-1)-alliance, then:

(i) If S is connected, one has:

$$\max\left\{\frac{-1+\sqrt{1+4n}}{2}, \frac{1+\sqrt{1+8m}}{4}\right\} \leq |S| \leq \min\left\{\frac{2n+1-\sqrt{4n+1}}{2}, \frac{m+3}{4}\right\}. \tag{1}$$

(ii) If S is not connected, the relation becomes:

$$\max\left\{\frac{1+\sqrt{4n-7}}{2}, \frac{5+\sqrt{8m-7}}{4}\right\} \leq |S| \leq \min\left\{\frac{2n+1-\sqrt{4n+1}}{2}, m\right\}. \tag{2}$$

Note that several examples have been presented in [26] for which these bounds are reached. For instance, all the bounds given in (1) are attained at the same time for the complete graph K_2, and the upper bound m given in (2) is reached when the graph G is constituted of not adjacent edges and every edge links a vertex of S with a vertex of $\bar{S}$.

Now, we summarize the results presented above by giving some bounds obtained for powerful k-alliance partition numbers in general and Cartesian product graphs. These results are given in Table 7.

Concluding remarks 6. As we can see from Table 7, and comparing with Table 6, we deduce that the powerful k-alliance partition numbers are studied on much less graph classes contrary to the powerful k-alliance numbers. Furthermore, we note that only the powerful partition numbers with index k namely $\psi_k^p(G)$ and $\psi_k^{gp}(G)$ that are investigated. Between these two parameters, the global powerful k-alliance partition number $\psi_k^{gp}(G)$ is the most studied one. Moreover, there are only two graph classes which are addressed in this case, for the general graphs only $\psi_k^{gp}(G)$ is studied and for the Cartesian product graphs both $\psi_k^p(G)$ and $\psi_k^{gp}(G)$ are studied.

5. Conclusion and discussion

Since the beginning of the last decade, when alliances in graphs were first introduced, much research has been focused on studying mathematical properties of various parameters of different types of k-alliances in graphs. In this paper, we have surveyed and discussed the principal known results obtained on defensive, offensive and powerful k-alliances by classifying them according to the different graph classes where the parameters are investigated. From this survey, we draw the following conclusions:

- By considering the classification criterion "graph class" in the study of the three kinds of k-alliances, we deduce that: the most studied graph classes on which there are more results are general, tree, planar and cartesian product graphs, and the least studied graph classes on which there are fewer results are cycle, path and line graphs.
- Several k-alliance numbers are defined in the literature. Some of them are received more attention and have been studied for various graph classes, such as $\gamma_{-1}^d(G)$, $\gamma_k^o(G)$ and $\gamma_{-1}^p(G)$. However, there are some parameters which are not studied for all graph classes, such as $A_1^o(G)$, $A_2^o(G)$ and $a_0^p(G)$, and other ones are not studied for certain graph classes, such as $A_k^d(G)$, $a_k^o(G)$ and $a_k^p(G)$.
- The k-alliance partition numbers have been studied on much less graph classes contrary to the k-alliance numbers. Moreover, only the partition numbers with index k are investigated in the case of partitioning of graphs into offensive (powerful) k-alliances.
- There are more studies and hence more results obtained on defensive k-alliances than on offensive (powerful) k-alliances.
- Some relationships are established between the global offensive k-alliance partition number and a coloration parameter namely the chromatic number. In this sense an extensive study which includes other parameters can be interesting.
- There are many investigations in the sense of theoretical aspects of k-alliances, but there are several prospects and progress to carry out in the algorithmic and computational side.
- The alliances with their important properties are used in interesting applications in several areas. As prospects, in practice there are many problems which have specific structures where the mathematical properties of the alliances can be involved and contribute to solve these problems.
- The definition of the defensive (-1)-alliance which takes into consideration the defense of a single vertex is generalized by Brigham et al. [53] to the concept of secure sets in order to forestall any attack on the entire alliance or any subset of the alliance. In this sense, it would be interesting to consider k-secure sets as extensions of defensive k-alliances and also to study the partitioning of graphs into k-secure sets. In this setting, motivations with practical examples would be needed.

Acknowledgments

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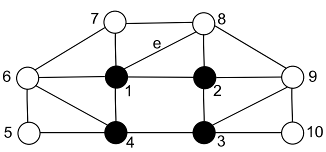
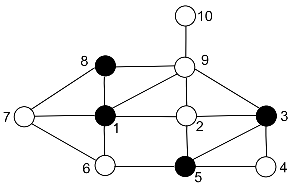
Appendix

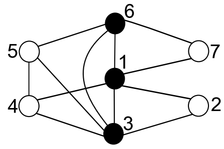
Note that the graphs given in Fig. 1 are denoted $G_1,\ G_2, \ldots, G_{15}$ respectively.



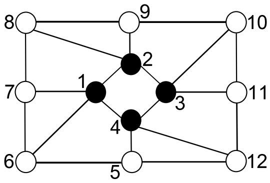
(a) $S_1 = \{1, 2, 3, 4\}$ is a (strong) global defensive alliance in $(G_1 - \{e\})$ G_1.



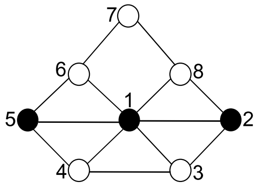
(b) $S_2 = \{1, 3, 5, 8\}$ is a (global) offensive alliance in $(G_2 - \{10\})$ G_2.



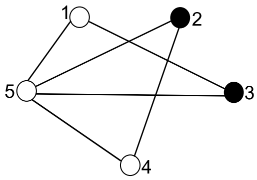
(c) $S_3 = \{1, 3, 6\}$ is a global powerful alliance in G_3.



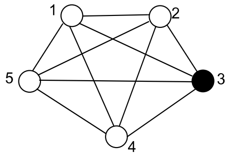
(d) $S_4 = \{1, 2, 3, 4\}$ is a global boundary defensive (resp. offensive) 0-alliance (resp. (-1)-alliance) in G_4.



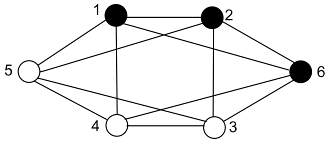
(e) $S_5 = \{1, 2, 5\}$ is a boundary offensive 1-alliance in G_5.



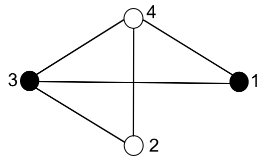
(f) $S_6 = \{2, 3\}$ is a global boundary powerful (-2)-alliance in G_6.



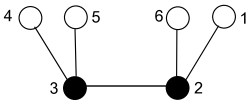
(g) $S_7 = \{3\}$ is a global defensive (-4)-alliance in G_7.



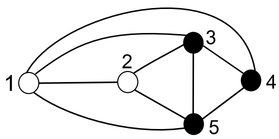
(h) $S_8 = \{1, 2, 6\}$ is a defensive 0-alliance in $G_8 = \mathcal{L}(K_4)$.



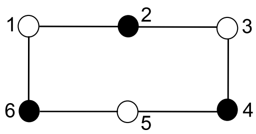
(i) $\{S_9 = \{1, 3\}, S'_9 = \{2, 4\}\}$ is a partition of G_9 into two defensive (-1)-alliances.



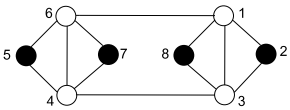
(j) $S_{10} = \{2, 3\}$ is a global offensive 1-alliance in G_{10}.



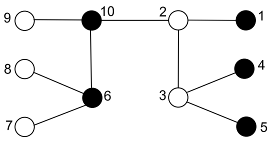
(k) $S_{11} = \{3, 4, 5\}$ is a global offensive 1-alliance in G_{11}.



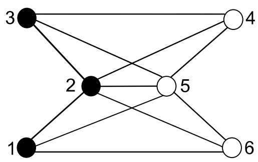
(l) $S_{12} = \{2, 4, 6\}$ is a global offensive k-alliance in $G_{12} = \mathcal{L}(C_6)$ with $k \in \{1, 2\}$.



(m) $S_{13} = \{2, 5, 7, 8\}$ is a global boundary powerful (-2)-alliance in G_{13}.



(n) $S_{14} = \{1, 4, 5, 6, 10\}$ is a global powerful (-1)-alliance in G_{14}.



(o) $\{S_{15} = \{1, 2, 3\}, S'_{15} = \{4, 5, 6\}\}$ is a partition of G_{15} into two global boundary powerful (-1)-alliances.

Fig. 1. Examples of k-alliances in graphs.

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