COP3502 1/16/24

⓪ P0 - can still submit until Wed

Dyn Mem Alloc - last week ✓

This Week - Algorithm Analysis + Math Background

Definitions of Big-Oh

$f(n) = O(g(n))$

$f(n) \in O(g(n))$

better — "is an element of"

Non-calc:

$\forall n > n_0 \; \exists c \in \mathbb{R}^+ \mid f(n) \le c \cdot g(n)$

Calc def

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, \quad c \text{ is a constant}$$

$3n^2 + 6n + 7 = O(n^2)$ ✓

$2^n + 7n = O(2^n)$

$\lg(3n^2 + 6n) = O(\lg n)$

all true tighter upper bounds

$n = O(n!)$ → not tight

is also true but not very meaningful

Intuitively, "f(n) is no bigger than g(n) within a constant multiplicative factor"

$f(n) = \Theta(g(n)) \leftrightarrow f(n) = O(g(n)) \wedge f(n) = \Omega(g(n))$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$$

$c \in$ Constant, $c > 0$.

$f(n) = \Omega(g(n)) \longrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$

"f(n) grows within a constant factor as fast as or more g(n)."

True: $3n^2+6n+7 = O(2n^2+5n+8)$

Standard is to write $O(f(n))$ in the most simple form:

$O(1), O(\lg n), O(\lg^2 n), O(\sqrt{n}), O(n), O(n\lg n), O(n\sqrt{n}), O(n^2), \ldots$
$O(2^n), O(n!)$

$\lg_b n = \dfrac{\lg_c n}{\lg_c b}$ } all logs w/ different bases > 1 are within const factor of each other

$\lg_c b$: Const

Why big-oh?

```
int res = 0;
for (int i = 0; i < n; i++)
    res += i;
```

many reasons why # stmts isn't perfectly fixed. variability
hardware
machine code
⋮

$O(n)$

```
for (int i = 0; i < n; i++)
    simple stmt
for (int i = 0; i < n; i++)
    simple stmt2
```

$O(n) + O(n)$

$O(n)$

```
for (int i = 0; i < n; i++)
    binsearch(arr, n, i)
```

n times

$O(\lg n)$

$O(n\lg n)$

loop $f(n)$ times:
  $g(n)$ work

$O(f(n) * g(n))$

```
int res=0;
for (int i=1; i<=n; i++)
    for (int j=1; j<=i; j++)
        res++;
```
runs differently each time!

$$1+2+3+4+\cdots+n$$

Summations

$$\sum_{i=a}^{b} f(i) = f(a)+f(a+1)+f(a+2)+\cdots+f(b)$$

$$\sum_{i=4}^{9} (2i-3) = 5+7+9+11+13+15$$

In code

```
sum=0
for (int i=a; i<=b; i++)
    sum += f(i);
```

$$\sum_{i=a}^{b} c = \underbrace{c+c+c\cdots c}_{b-a+1 \text{ times}} = c(b-a+1)$$

$$S = \sum_{i=1}^{n} i = 1 \xrightarrow{+1} 2+3+\cdots+n$$

$$S = \underset{n}{100} \xrightarrow{-1} + \underset{(n-1)}{99} + \underset{(n-2)}{98} + \cdots + 1$$

$$2S = \underset{(n+1)}{101} + \underset{(n+1)}{101} + \underset{(n+1)}{101} + \cdots \; (n+1)$$

$$2S = n(n+1) \implies \boxed{S = \frac{n(n+1)}{2}}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=a}^{b} c \cdot f(i) = c \sum_{i=a}^{b} f(i)$$

$$\sum_{i=a}^{b} \left(f(i)+g(i)\right) = \sum_{i=a}^{b} f(i) + \sum_{i=a}^{b} g(i)$$

$$\begin{aligned}
\sum_{i=1}^{n} (3i+2) &= \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 2 \\
&= 3\sum_{i=1}^{n} i + 2n \\
&= 3 \cdot \frac{n(n+1)}{2} + 2n \\
&= \frac{3n(n+1)+4n}{2} \\
&= \frac{n(3(n+1)+4)}{2} \\
&= \boxed{\frac{n(3n+7)}{2}}
\end{aligned}$$

$$\sum_{i=a}^{b} f(i) = \sum_{i=1}^{b} f(i) - \sum_{i=1}^{a-1} f(i)$$

$b > a > 1$

$f(a) + f(a+1) + f(a+2) + \cdots \quad f(b)$

$\boxed{f(1) + f(2) + \cdots \quad f(a-1)} + \boxed{f(a) + f(a+1) + \cdots \quad f(b)}$

$$\sum_{i=2n+1}^{n^2} i = \sum_{i=1}^{n^2} i - \sum_{i=1}^{2n} i$$

$$= \frac{n^2(n^2+1)}{2} - \frac{2n(2n+1)}{2}$$

$$= \frac{n^4+n^2-4n^2-2n}{2}$$

$$= \frac{n^4-3n^2-2n}{2}$$

$$= \frac{n(n^3-3n-2)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Sequence

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

a_1 = 1st term r = common ratio.

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots, \quad |r|<1$$

$$- \quad rS = a_1 r + a_1 r^2 + a_1 r^3 + \cdots$$

$$S - rS = a_1$$

$$S(1-r) = a_1 \longrightarrow S = \frac{a_1}{1-r}$$

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}}$$

$|x|<1$

$$\overbrace{S = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots \ \cancel{a_1 r^{n-1}}}^{n \text{ terms}}$$

$$-rS = \qquad \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots \ \cancel{a_1 r^{n-1}} + a_1 r^n \qquad \sum_{i=0}^{n-1} a_1 \cdot r^i$$

$r \neq 1$

$$S - rS = a_1 \qquad - a_1 r^n$$

$$S(1-r) = a_1(1-r^n)$$

$$\boxed{S = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(r^n-1)}{r-1} = \sum_{i=0}^{n-1} a_1 \cdot r^i}$$

$$S = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + 5 \times \frac{1}{32} + \cdots$$

$$-\frac{1}{2}S = \qquad 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} + 4 \times \frac{1}{32} \qquad \sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

$$\frac{1}{2}S = \frac{\frac{1}{2} \rightarrow a_1}{1 - \frac{1}{2} \rightarrow r}$$

$$\frac{1}{2}S = \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$\boxed{S = 2}$$

```
res = 0;
for (int i = 1; i <= n; i++)
    for (int j = 1; j <= i; j++)
        res++;
```

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = O(n^2)$$

```
for (int i = 1; i <= n; i++)
    binsearch(arr, i, value);
```

$$\lg 1 + \lg 2 + \lg 3 + \dots = \sum_{i=1}^{n} \lg i = \lg n!$$

$\lg a + \lg b = \lg ab$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Stirling's Approx

big term

$$\lg \left(\frac{n}{e}\right)^n$$

$$= n \lg \left(\frac{n}{e}\right)$$

$$= n(\lg n - \lg e)$$

$$= n\lg n - n\lg e$$

$$= O(n \lg n)$$

Alg A run-time $O(n^2)$

$n = 50{,}000$ took $\cancel{380}$ 400 ms

how long will the alg take on an input of size $n = 150{,}000$?

$T(n) = cn^2$ for some const c. (Slight simplification)

$T(50000) = c(50000)^2 = 400\,ms$

$$c = \frac{400\,ms}{(50000)^2}$$

$$T(150{,}000) = c(150000)^2$$

$$= \frac{400\,ms}{(50000)^2} \times (150000)^2$$

$$= 400\,ms \times \left(\frac{150000}{50000}\right)^2$$

$$= 400\,ms \times 3^2$$

$$= \boxed{3.6 \text{ seconds}}$$

$O(rc^2)$ $\quad r=200,\ c=500 \implies 5.0\ \text{sec}$

$r=800,\ c=300$

$T(r,c) = K r c^2$ for some const K

$T(200,500) = K(200)(500)^2 = 5\ \text{sec}$

$$K = \frac{5\ \text{sec}}{200(500)^2}$$

$$T(800,300) = \frac{5\ \text{sec}}{\cancel{200}(500)^2} \times \cancelto{4}{800} \times 300^2$$

$$= 20\ \text{sec} \times \left(\frac{300}{500}\right)^2$$

$$= \cancelto{4}{20}\ \text{sec} \times \frac{9}{\cancelto{5}{25}}$$

$$= \frac{36}{5}\ \text{sec}$$

$$= \boxed{7.2\ \text{sec}}$$