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Brief Paper

Robust Iterative Learning Control for a Class of Nonlinear Systems*

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Key Words—Iterative learning control; robust nonlinear control; multi-input multi-output system.

Abstract—A novel nonlinear control scheme—robust iterative learning control (RILC) is developed in this paper. The new robust ILC system provides a general framework targeting at synthesizing learning control and robust control methods with the help of Lyapunov's direct method, thereafter being able to handle more general classes of nonlinear uncertain systems. In the proposed control scheme, learning control and variable structure control are made to function in a complementary manner. The nonlinear learning control strategy is applied directly to the structured system uncertainties which can be separated and expressed as products of unknown state-independent functions and known state-dependent functions. For non-structured system uncertainties associated with known bounding functions as the only *a priori* knowledge, variable structure control (VSC) strategy is applied to ensure the global asymptotic stability. In addition, important issues regarding the objective trajectory categories, resetting condition, derivative signal requirement and their relationships have been made clear in this paper. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

Nowadays intelligent control and robust control are the two main trends in the area of control theories and technologies. On one hand, *learning* or adaptive capability is the main characteristic of intelligent control. On the other hand, robustness or insensitivity is the main characteristic of the robust control. Learning is an *active* way to address system uncertainties in the sense that it tries to identify the uncertain source such that the *optimal* control arrangement can be made to eliminate the uncertain effects. On the contrary, robust control works as a *passive* way to deal with system uncertainties in the sense that it tries to estimate the worst situation such that the *safest* control arrangement can be made to protect the control system from the uncertain source. Whether using intelligent control or robust control is highly dependent upon the available information concerning control environment and control objectives. The ultimate target of this paper is to explore the possibility of synthesizing both learning and robust control strategies to generate new control system which can easily fulfill control objectives that are impossible for either learning control or robust control alone to handle.

Most learning control methods can only incorporate simple P or PD-types control, or even work in open loop such as most ILC based algorithms (Arimoto *et al*., 1984; Oh *et al*., 1988; Saab, 1995; Xu, 1997). The simplicity of the existing learning control algorithms, or the lack of effective analysis approaches for complicated learning control algorithms in a large degree limits the extension of learning control to more general classes of nonlinear uncertain systems. Moreover, the underlying open-loop nature of many existing learning control methods further deteriorate the system transient responses for each operation cycle. On the other hand, robust control methods such as VSC operate in closed loop and allow highly nonlinear, complicated control algorithms to be used with the help of Lyapunov techniques. It is well known that, VSC method works well when the size of system uncertainties are limited by certain known bounding functions, no matter those uncertainties are parametric or structural, periodic or non-periodic, state-dependent or exogenous ones (Utkin, 1978). However, VSC fails to work if those bounding functions are not available. On the contrary, most learning control methods do not require system knowledge regarding uncertain bounds, but only apply to limited classes of nonlinear uncertain dynamics which must be periodic. Hence, by incorporating robust control methods such as VSC together with Lyapunov techniques into learning control systems, one may expect that much boarder classes of nonlinear uncertain system can be handled.

From the point of view of system performance, learning control methods only ensure the convergence of the control system with respect to the repeated operations, whereas robust control methods only guarantee the convergence of the control system along the time horizon. By synthesizing learning control and robust control methods, we can retain the advantage of both types of convergence. For instance, we can design a VSC to suppress bounded non-periodic disturbances for each of the operation cycles. Then the learning control part can be arranged to eliminate periodic uncertainties gradually through repeated operations. In this way the stable controller design becomes much easier because we now have two degrees of freedom: stabilizing the system either in time domain or in terms of iterations. As a consequence, more general classes of systems can be easily coped with.

Compared with robust control methods, the main advantage of synthesizing learning control and robust control is its capability of improving the system performance gradually with respect to periodic operations or repeatable control tasks with a fixed finite period. In the proposed robust learning control scheme, the contribution from learning control part is to *learn* and eliminate state-independent uncertainties as much as possible. The contribution from the robust part is to suppress the remaining system uncertainties in which only the upper bounds are available for design.

It shows, that quite general classes of nonlinear dynamic systems with high uncertainties can be easily dealt with by the new control scheme. The robust learning control system possesses the capability of working in either iterative or repetitive control mode for different control objectives. Through analyzing the developed control system in a systematic way, important issues regarding the objective trajectory categories, resetting condition, derivative signal requirement and their relationships have been made clear.

This paper is organized as follows. Section 2 describes the control problem and objectives. Section 3 details the design and analysis of the proposed robust learning control scheme with

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resetting condition. Section 4 provides an example with simulation results.

2. *Problem statement*

Consider a higher order MIMO nonlinear dynamical system described by

$$\dot{\mathbf{x}}_i = \mathbf{x}_{i+1}, \quad i = 1, \dots, m-1, \qquad (1)$$
$$\dot{\mathbf{x}}_m = \mathbf{h}(\mathbf{x}, \mathbf{p}, t) + \mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega) + B(\mathbf{z}, \mathbf{p}, t)\mathbf{u},$$

where $\mathbf{x}_i \in \mathscr{R}^{n\times 1}$, $i = 1, \dots, m$; $\mathbf{x} \triangleq [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_m^T]^T \in \mathscr{X} \subseteq \mathscr{R}^{nm\times 1}$ is the measurable state vector of the system. $\mathbf{u} \in \mathscr{R}^{n\times 1}$ is the control input vector of the system. $\mathbf{z} \in \mathscr{Z} \subseteq \mathscr{R}^{q\times 1}$ where $\mathscr{Z}$ is a subset of the state space $\mathscr{X}$ with dimension $q \le nm$. $\mathbf{p} \in \mathscr{P}$ is an unknown system parameter vector. $\mathscr{P}$ is the set of admissible system parameters. $\mathbf{h}(\mathbf{x}, \mathbf{p}, t)$ represents the structured uncertainties and $\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega)$ represents unstructured uncertainties. ω represents any aperiodic factor such as aperiodic exogenous disturbance or system noise. $B(\mathbf{z}, \mathbf{p}, t) \in \mathscr{R}^{n\times n}$ is the input distribution matrix. In this paper, we make the following assumptions on vectors $\mathbf{h}$, $\mathbf{g}$ and matrix B.

Assumption A1. Each element of the unknown function vector $\mathbf{h}(\mathbf{x}, \mathbf{p}, t)$ can be expressed as

$$h_i(\mathbf{x}, \mathbf{p}, t) = \boldsymbol{\theta}_i^T(\mathbf{p}, t)\boldsymbol{\xi}_i(\mathbf{x}, t), \quad i = 1, 2, \dots, n, \qquad (2)$$

where $\boldsymbol{\theta}_i^T = [\theta_i^1 \cdots \theta_i^{l_i}]$ is the unknown function vector of $\mathbf{p}$ and t; and $\boldsymbol{\xi}_i^T = [\xi_i^1 \cdots \xi_i^{l_i}]$ is the known function vector of $\mathbf{x}$ and t.

Assumption A2. The nonlinear function vector $\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega)$ is bounded such that

$$\forall t \in [0,\ T_f] \quad \forall \mathbf{x} \in \mathscr{X} \quad \forall \mathbf{p} \in \mathscr{P},$$
$$d_{i,\min}(\mathbf{x}, t) \le d_i(\mathbf{x}, \mathbf{p}, t) \le d_{i,\max}(\mathbf{x}, t), \quad i = 1, 2, \dots, n,$$

where $d_i(\mathbf{x}, t)$ is the ith element of the function vector $\mathbf{d}$. $d_{i,\min}(\mathbf{x}, t)$ and $d_{i,\max}(\mathbf{x}, t)$ are known and continuous bounding functions with respect to $\mathbf{x}$ and t.

Assumption A3. Matrix $B(\mathbf{z}, \mathbf{p}, t)$ is positive definite for all $t \in [0, T_f]$, $\mathbf{z} \in \mathscr{Z}$, $\mathbf{p} \in \mathscr{P}$, and satisfies the following inequalities:

$$0 < \lambda_{\min} I \le B,$$
$$\left\| \frac{\partial B}{\partial z_i} \right\| \le |\lambda_{i,\max}|, \qquad (3)$$
$$i = 0, 1, \dots, q,$$

where $z_0 \triangleq t$; the inequality $A_1 \le A_2$ is defined as $\lambda_{\max}(A_1) \le \lambda_{\min}(A_2)$. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and the minimum eigenvalues, respectively. Each element of the matrix B^{-1} can be expressed as

$$\boldsymbol{\phi}_{ij}^T(\mathbf{p}, t)\boldsymbol{\eta}_{ij}(\mathbf{z}, t), \quad i, j = 1, 2, \dots, n,$$

where $\boldsymbol{\phi}_{ij}^T = [\phi_{ij}^1 \cdots \phi_{ij}^{l_{ij}}]$ is the unknown function vector of $\mathbf{p}$ and t; $\eta_{ij}^T = [\eta_{ij}^1 \cdots \eta_{ij}^{l_{ij}}]$ is the vector of known nonlinear functions of $\mathbf{z}$ and t.

In this paper, $\|A\|$ with respect to a square matrix A is the induced matrix norm defined as

$$\|A\| = \sup\left\{\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \text{ for } \mathbf{x} \ne 0\right\}. \qquad (4)$$

Since $\|\cdot\|$ is the Euclidean norm for vectors, the corresponding induced matrix norm is

$$\|A\| = [\lambda_{\max}(A^T A)]^{1/2}$$

for a real matrix A, and

$$\|A\| = |\lambda_{\max}(A)|$$

for a real symmetric matrix A.

The control objective is to find an appropriate control input $\mathbf{u} \in \mathscr{R}^{n\times 1}$ for the uncertain nonlinear system (1) such that the system state $\mathbf{x}(t)$ follows $\mathbf{x}_d(t)$ with a prescribed accuracy ε as follows:

$$\forall t \, \|\mathbf{x}_d(t) - \mathbf{x}(t)\| \le \varepsilon, \qquad (5)$$

where $\mathbf{x}_d = [\mathbf{x}_{1,d}^T \cdots \mathbf{x}_{m,d}^T]^T$ is the desired state trajectory.

In this article the following two categories of trajectory $\mathbf{x}_d$ are under consideration:

Category I. A desired state trajectory $\mathbf{x}_{1,d}(t)$, which is defined on a finite interval of time $[0, T_f]$, is differentiable with respect to t up to the mth order and all its higher-order derivatives

$$\mathbf{x}_{1,d}^{(i)}(t) \triangleq \mathbf{x}_{i+1,d}(t), \quad i = 0, \dots, m$$

are available over $t \in [0, T_f]$.

Category II. In addition to Category I, the desired trajectory $\mathbf{x}_d(t)$ of Category II satisfies the following alignment condition:

$$\mathbf{x}_d(0) = \mathbf{x}_d(T_f).$$

Remark 1. The desired state trajectory of Category I is more general in the sense that it does not require the alignment of the initial value $\mathbf{x}_d(0)$ and the terminal value $\mathbf{x}_d(T_f)$. As will be shown later, the robust iterative control strategy with resetting can be applied to both Categories I and II, whereas iterative strategy without resetting can only be applied to Category II.

3. *Robust iterative learning control scheme*

The underlying idea of robust iterative learning control is to *learn* and approach the unknown state-independent functions. Learning mechanism is designed to identify all those state-independent components and leave the remaining unknowns to the robust control. Lyapunov's direct method is applied such that variable structure control strategy can be incorporated to guarantee, for nonlinear and uncertain dynamic systems, the global asymptotic convergence with respect to the iterations.

3.1. *Modeling of inverse dynamics.* To distinguish the required control efforts for learning and robust control, respectively, the inverse model of the nonlinear plant (1) is needed. First, for mth-order dynamic systems, it is necessary to define an extended tracking error, which is in fact a switching surface

$$\boldsymbol{\sigma}(t) = \sum_{i=1}^{m} \alpha_i[\mathbf{x}_{i,d}(t) - \mathbf{x}_i(t)], \quad \alpha_m = 1, \qquad (6)$$

in which $\alpha_i\ (i = 1, \dots, m)$ are coefficients of a Hurwitz polynomial.

Taking derivative of $\boldsymbol{\sigma}$ with respect to time t yields

$$\dot{\boldsymbol{\sigma}}(t) = \sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,d} - \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1} - \mathbf{h}(\mathbf{x}, \mathbf{p}, t) - \mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega) - B(\mathbf{z}, \mathbf{p}, t)\mathbf{u}, \qquad (7)$$

where $\mathbf{x}_{m+1,d} = \dot{\mathbf{x}}_{m,d}$. In order to partition the system uncertainties into the known state-dependent functions, unknown state-independent functions and unknown state-dependent functions with known bounds, rearrange the above error dynamics as follows:

$$\mathbf{u} = B^{-1}(\mathbf{z}, \mathbf{p}, t)\left[\sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,d} - \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1} - \mathbf{h}(\mathbf{x}, \mathbf{p}, t)\right] + B^{-1}(\mathbf{z}, \mathbf{p}, t)\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega) - B^{-1}(\mathbf{z}, \mathbf{p}, t)\dot{\boldsymbol{\sigma}}(t)$$
$$= \mathbf{r} + \mathbf{g} - B^{-1}\dot{\boldsymbol{\sigma}}, \qquad (8)$$

where

$$\mathbf{r} = B^{-1}(\mathbf{z}, \mathbf{p}, t)\left[\sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,d} - \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1} - \mathbf{h}(\mathbf{x}, \mathbf{p}, t)\right] \qquad (9)$$

represents all the nonlinear terms which can be separated into the state-dependent functions and unknown state-independent functions, and

$$\mathbf{g} = -B^{-1}(\mathbf{z}, \mathbf{p}, t)\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega) \qquad (10)$$

represents inseparable nonlinear terms with known bounding functions.

Remark 2. The linear term

$$\sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,\mathrm{d}} - \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1},$$

in equation (9) is completely known, hence its bounding function can be easily calculated. On the other hand, this linear term consists of known state-dependent functions (constants and state variables) and known state-independent time functions. Therefore, both robust control and learning methods could apply. However, in order to maximize the learning control capability and reduce the robust control efforts, arrangement as equation (9) is preferred.

3.2. *Function partition.* Based on the inverse dynamics expression, it is possible now to partition function vector $\mathbf{r}$ in equation (9) to facilitate the design of learning mechanism. From Assumptions A1 and A3, vector $\mathbf{r}$ can be expressed by

$$\mathbf{r} = A(\mathbf{x}, t)\gamma(\mathbf{x}_\mathrm{d}, \mathbf{p}, t), \tag{11}$$

where $A(\mathbf{x}, t)$ is a known matrix with appropriate dimensions. $\gamma(\mathbf{x}_d, \mathbf{p}, t)$ is an unknown function vector of time t and unknown parameters $\mathbf{p}$. Note that $\mathbf{x}_\mathrm{d}(t)$ is a known function vector of time t only. In details, we have

$$\begin{aligned} A(\mathbf{x}, t) &= \mathrm{diag}(\boldsymbol{\zeta}_1^\mathrm{T}, \ldots, \boldsymbol{\zeta}_n^\mathrm{T}), \\ \boldsymbol{\zeta}_i^\mathrm{T} &= [\boldsymbol{\eta}_{i,1}^\mathrm{T} \cdots \boldsymbol{\eta}_{i,n}^\mathrm{T} \,\vdots\, \boldsymbol{\eta}_{i,1}^\mathrm{T}\pi_1 \cdots \boldsymbol{\eta}_{i,n}^\mathrm{T}\pi_n \,\vdots\, \boldsymbol{\eta}_{i,1}^\mathrm{T}\xi_1^1 \cdots \boldsymbol{\eta}_{i,1}^\mathrm{T}\xi_1^{l_1} \,\vdots \\ &\quad \boldsymbol{\eta}_{i,2}^\mathrm{T}\xi_2^1 \cdots \,\vdots\, \boldsymbol{\eta}_{i,n}^\mathrm{T}\xi_n^1 \cdots \eta_{i,n}^\mathrm{T}\xi_n^{l_n}], \\ \boldsymbol{\pi} &= [\pi_1 \cdots \pi_n]^\mathrm{T} \triangleq \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1}. \end{aligned} \tag{12}$$

and correspondingly we have

$$\begin{aligned} \gamma^\mathrm{T} &= [\gamma_1^\mathrm{T} \cdots \gamma_n^\mathrm{T}], \\ \gamma_i^\mathrm{T}(\mathbf{x}_\mathrm{d}, \mathbf{p}, t) &= [\boldsymbol{\phi}_{i,1}^\mathrm{T}\kappa_1 \cdots \boldsymbol{\phi}_{i,n}^\mathrm{T}\kappa_n \,\vdots\, -\boldsymbol{\phi}_{i,1}^\mathrm{T} \cdots -\boldsymbol{\phi}_{i,n}^\mathrm{T} \,\vdots \\ &\quad \boldsymbol{\phi}_{i,1}^\mathrm{T}\theta_1^1 \cdots \phi_{i,1}^\mathrm{T}\theta_1^{l_1} \,\vdots\, \boldsymbol{\phi}_{i,2}^\mathrm{T}\theta_2^1 \cdots \,\vdots\, \boldsymbol{\phi}_{i,n}^\mathrm{T}\theta_n^1 \cdots \boldsymbol{\phi}_{i,n}^\mathrm{T}\theta_n^{l_n}], \\ \boldsymbol{\kappa} &= [\kappa_1 \cdots \kappa_n]^\mathrm{T} \triangleq \sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,\mathrm{d}}. \end{aligned} \tag{13}$$

Note that the product of the row vector $\boldsymbol{\zeta}_i^\mathrm{T}$ and column vector γ_i represents the product of the ith column vector of B^{-1} and the vector

$$\left[\sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,\mathrm{d}} - \sum_{i=1}^{m-1} \alpha_i \mathbf{x}_{i+1} - \mathbf{h}(\mathbf{x}, \mathbf{p}, t)\right].$$

in equation (9). The partition is arranged in such way that all state-relevant terms are assigned to the matrix A, and the remaining to the vector γ which is to be learned through iterations. Here we again have the flexibility of assigning the known

$$\sum_{i=1}^{m} \alpha_i \mathbf{x}_{i+1,\mathrm{d}}$$

either to A or to γ. For simplicity of controller construction and computation, we assign all $\mathbf{x}_\mathrm{d}$ related terms to γ in the proposed method.

The error dynamics (8) can then be rewritten as follows:

$$\begin{aligned} \mathbf{u} &= \mathbf{r} + \mathbf{g} - B^{-1}\dot{\boldsymbol{\sigma}} \\ &= A(\mathbf{x}, t)\gamma(\mathbf{x}_\mathrm{d}, \mathbf{p}, t) + \mathbf{g}(\mathbf{x}_\mathrm{d}, \mathbf{x}, \mathbf{p}, t, \omega) - B^{-1}(\mathbf{x}, \mathbf{p}, t)\dot{\boldsymbol{\sigma}}. \end{aligned} \tag{14}$$

3.3. *Bounding function calculation.* To design variable structure control it is necessary to find the bounding function of the vector $\mathbf{g}$ which are functions of $\mathbf{x}_\mathrm{d}$, $\mathbf{x}$, $\mathbf{p}$, t and ω. Taking Euclidean norm on both sides of equation (10),

$$\|\mathbf{g}\| \le \|B^{-1}(\mathbf{z}, \mathbf{p}, t)\| \cdot \|\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega)\|. \tag{15}$$

From Assumption A2

$$\begin{aligned} \|\mathbf{d}(\mathbf{x}, \mathbf{p}, t, \omega)\| &\le l_\mathrm{d}(\mathbf{x}, t) \\ &\triangleq \left[\sum_{i=1}^{n} (\max\{|d_{i,\min}|, |d_{i,\max}|\})^2\right]^{1/2}. \end{aligned} \tag{16}$$

From Assumption A3, we have

$$0 \le B^{-1} \le \lambda_{\min}^{-1} I$$

and hence

$$\|B^{-1}(\mathbf{z}, \mathbf{p}, t)\| \le \max_i |\lambda_i(B^{-1})| \le |\lambda_{\min}^{-1}| \triangleq l_\mathrm{B}(\mathbf{z}, t) \tag{17}$$

Finally, the bounding function of the system uncertainty $\mathbf{g}$ is

$$\begin{aligned} \|\mathbf{g}\| &\le l_\mathrm{g}(\mathbf{x}_\mathrm{d}, \mathbf{x}, t) \\ &\triangleq l_\mathrm{B}(\mathbf{z}, t) l_\mathrm{d}(\mathbf{x}, t). \end{aligned} \tag{18}$$

3.4. *Robust ILC algorithm.* The iterative type robust learning control input consists of two parts

$$\begin{aligned} \mathbf{u}_j &= A_j \mathbf{v}_j + \mathbf{w}_j, \\ A_j &\triangleq A(\mathbf{x}_j, t), \end{aligned} \tag{19}$$

where j indicates the number of the learning trial. $\mathbf{v}_j$ is the recursive control part as follows

$$\mathbf{v}_j = \mathbf{v}_{j-1} + \beta_1 A_{j-1}^\mathrm{T} \boldsymbol{\sigma}_{j-1} \tag{20}$$

and β_1 is the learning control gain. $\mathbf{w}_j$ is the robust control part which can be decided through minimizing the difference of the following Lyapunov function

$$V_j = \int_0^{T_\mathrm{f}} \|\gamma(\tau) - \mathbf{v}_j(\tau)\|^2 \,\mathrm{d}\tau, \tag{21}$$

which consists of Euclidean norm and $\mathscr{L}_2$ norm of learning error $\gamma(t) - \mathbf{v}_j(t)$. The difference of Lyapunov function between two successive trials is

$$\begin{aligned} \Delta V_j &= V_{j+1} - V_j \\ &= \int_0^{T_\mathrm{f}} [\|\gamma(\tau) - \mathbf{v}_{j+1}(\tau)\|^2 - \|\gamma(\tau) - \mathbf{v}_j(\tau)\|^2] \,\mathrm{d}\tau. \end{aligned} \tag{22}$$

3.5. *Derivation of Robust control law.* Substituting the learning law (20) into equation (22) yields

$$\begin{aligned} \Delta V_j &= \int_0^{T_\mathrm{f}} \{[\gamma(\tau) - \mathbf{v}_j(\tau) - \beta_1 A_j^\mathrm{T}\boldsymbol{\sigma}_j]^\mathrm{T}[\gamma(\tau) - \mathbf{v}_j(\tau) - \beta_1 A_j^\mathrm{T}\boldsymbol{\sigma}_j] \\ &\quad - [\gamma(\tau) - \mathbf{v}_j(\tau)]^\mathrm{T}[\gamma(\tau) - \mathbf{v}_j(\tau)]\} \,\mathrm{d}\tau \\ &= \int_0^{T_\mathrm{f}} [\beta_1^2 \boldsymbol{\sigma}_j^\mathrm{T} A_j A_j^\mathrm{T} \boldsymbol{\sigma}_j - 2\beta_1 \boldsymbol{\sigma}_j^\mathrm{T} A_j(\gamma - \mathbf{v}_j)] \,\mathrm{d}\tau. \end{aligned} \tag{23}$$

On the other hand, from the inverse dynamics (14) and the control law (9) we have

$$\begin{aligned} \mathbf{u}_j &= \mathbf{r}_j + \mathbf{g}_j - B_j^{-1}\dot{\boldsymbol{\sigma}}_j = A_j\gamma + \mathbf{g}_j - B_j^{-1}\dot{\boldsymbol{\sigma}}_j \\ &= A_j \mathbf{v}_j + \mathbf{w}_j \end{aligned} \tag{24}$$

or

$$A_j(\gamma - \mathbf{v}_j) = \mathbf{w}_j - \mathbf{g}_j + B_j^{-1}\dot{\boldsymbol{\sigma}}_j.$$

Substituting above relation into equation (23) we have

$$\begin{aligned} \Delta V_j &= \int_0^{T_\mathrm{f}} \{\beta_1^2 \boldsymbol{\sigma}_j^\mathrm{T} A_j A_j^\mathrm{T} \boldsymbol{\sigma}_j - 2\beta_1 \boldsymbol{\sigma}_j^\mathrm{T}[\mathbf{w}_j - \mathbf{g}_j + B_j^{-1}\dot{\boldsymbol{\sigma}}_j]\} \mathrm{d}\tau \\ &= \int_0^{T_\mathrm{f}} \{\beta_1^2 \boldsymbol{\sigma}_j^\mathrm{T} A_j A_j^\mathrm{T} \boldsymbol{\sigma}_j - 2\beta_1 \boldsymbol{\sigma}_j^\mathrm{T}[\mathbf{w}_j - \mathbf{g}_j + \tfrac{1}{2} B_j^{-1}\dot{B}_j B_j^{-1}\boldsymbol{\sigma}_j]\} \,\mathrm{d}\tau \\ &\quad - \beta_1 \boldsymbol{\sigma}_j^\mathrm{T} B_j^{-1} \boldsymbol{\sigma}_j |_0^{T_\mathrm{f}} \\ &\le -\beta_1 \boldsymbol{\sigma}_j^\mathrm{T}(T_\mathrm{f}) B_j^{-1}(T_\mathrm{f}) \boldsymbol{\sigma}_j(T_\mathrm{f}) + \beta_1 \boldsymbol{\sigma}_j^\mathrm{T}(0) B_j^{-1}(0) \boldsymbol{\sigma}_j(0) \\ &\quad + \int_0^{T_\mathrm{f}} \{\beta_1^2 \boldsymbol{\sigma}_j^\mathrm{T} A_j A_j^\mathrm{T} \boldsymbol{\sigma}_j - 2\beta_1 \boldsymbol{\sigma}_j^\mathrm{T} \mathbf{w}_j + 2\beta_1 \|\boldsymbol{\sigma}_j\| \cdot \|\mathbf{g}_j\| \\ &\quad + \beta_1 \|B_j^{-1}\dot{B}_j B_j^{-1}\| \cdot \|\boldsymbol{\sigma}_j\|^2\} \,\mathrm{d}\tau. \end{aligned} \tag{25}$$

According to Assumption A3, it is easy to calculate the upper bound of $B_j^{-1}\dot{B}_jB_j^{-1}$. Because

$$B_j^{-1}\dot{B}_jB_j^{-1} = B_j^{-1}\sum_{i=0}^{q}\frac{\partial B_j}{\partial z_i}\dot{z}_iB_j^{-1}, \tag{26}$$

from matrix norm property we have

$$\begin{aligned}\|B_j^{-1}\dot{B}_jB_j^{-1}\| &\le \|B_j^{-1}\|^2\cdot\|\dot{B}_j\|\\ &\le \|B_j^{-1}\|^2\sum_{i=0}^{q}\left\|\frac{\partial B_j}{\partial z_i}\right\|\cdot|\dot{z}_i|\\ &\le |\lambda_{\min}^{-1}|^2\sum_{i=0}^{q}|\lambda_{i,\max}|\cdot|\dot{z}_i| \triangleq \rho_j.\end{aligned} \tag{27}$$

Notice that, in the presence of the aperiodic functions $\mathbf{d}$, control input $\mathbf{w}_j$ may not be zero even if the state error $\mathbf{x}_d - \mathbf{x}$ or the quantity $\boldsymbol{\sigma}$ approaches zero. This means that an theoretically infinitely high control gain with respect to $\boldsymbol{\sigma}$ may be needed near the vicinity of the switching surface $\boldsymbol{\sigma} = 0$. It is well known that variable structure control in sliding mode possesses the high-gain property across the switching surface and yet retaining finite control authority. Therefore we choose VSC as an effective robust approach for such cases. To make ΔV_j as negative as possible, the robust control part $\mathbf{w}_j$ is designed as follows:

$$\mathbf{w}_j = \frac{\beta_1}{2}A_jA_j^{\mathrm{T}}\boldsymbol{\sigma}_j + l_g\,\mathbf{sgn}(\boldsymbol{\sigma}_j) + (\rho_j + \beta_{\mathrm{fb}})\boldsymbol{\sigma}_j, \tag{28}$$

where β_{fb} is a constant feedback gain, and

$$\mathbf{sgn}(\boldsymbol{\sigma}_j) \triangleq [\mathrm{sgn}(\sigma_1), \ldots, \mathrm{sgn}(\sigma_n)]^{\mathrm{T}}.$$

Substituting equation (28) into equation (26) and notice

$$\boldsymbol{\sigma}_j^{\mathrm{T}}\,\mathbf{sgn}(\boldsymbol{\sigma}_j) \ge \|\boldsymbol{\sigma}_j\|,$$

we have

$$\begin{aligned}\Delta V_j \le &-\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(T_f)B_j^{-1}(\mathbf{z}(T_f), T_f)\boldsymbol{\sigma}_j(T_f)\\ &+\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(0)B_j^{-1}(\mathbf{z}(0), 0)\boldsymbol{\sigma}_j(0)\\ &-2\int_0^{T_f}\beta_1\beta_{\mathrm{fb}}\|\boldsymbol{\sigma}_j\|^2\,\mathrm{d}\tau.\end{aligned} \tag{29}$$

Based on the inequality (29) we can make the following conclusions.

Theorem 1 (*Robust ILC with resetting*). Assume that the alignment of the initial states $\mathbf{x}_j(0) = \mathbf{x}_{j,d}(0)$ is available for all trials, the learning control law (20) and robust control law (28) guarantee that the nonlinear system (1) tracks the given trajectory of Category I asymptotically while all state variables are globally and uniformly bounded.

Proof. Under the resetting condition,

$$\boldsymbol{\sigma}_j^{\mathrm{T}}(0)B_j^{-1}(0)\boldsymbol{\sigma}_j(0) = 0, \quad j = 0, 1, \ldots .$$

Therefore, it follows from the relationship (29) that

$$\Delta V_j \le -\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(T_f)B_j^{-1}(T_f)\boldsymbol{\sigma}_j(T_f) - 2\beta_1\beta_{\mathrm{fb}}\int_0^{T_f}\|\boldsymbol{\sigma}_j\|^2\,\mathrm{d}\tau, \tag{30}$$

which is negative definite when $\boldsymbol{\sigma}_j(t) \ne 0$, $t \in [0, T_f]$. Now taking summation of ΔV_j up to k yields

$$\begin{aligned}\sum_{j=0}^{k}\Delta V_j &= V_{k+1} - V_0\\ &\le -\sum_{j=0}^{k}\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(T_f)B_j^{-1}(\mathbf{z}(T_f), T_f)\boldsymbol{\sigma}_j(T_f)\\ &\quad -\sum_{j=0}^{k}2\beta_1\beta_{\mathrm{fb}}\int_0^{T_f}\|\boldsymbol{\sigma}_j\|^2\,\mathrm{d}\tau.\end{aligned} \tag{31}$$

Consequently, we have

$$\sum_{j=0}^{k}\int_0^{T_f}\|\boldsymbol{\sigma}_j\|^2\,\mathrm{d}\tau \le V_0/(2\beta_1\beta_{\mathrm{fb}}) < \infty. \tag{32}$$

Taking the limit of $k \to \infty$ leads to

$$\lim_{k\to\infty}\int_0^{T_f}\|\boldsymbol{\sigma}_k\|^2\,\mathrm{d}\tau = 0.$$

Since the switching surface (6) is selected to be *Hurwitz*, $\boldsymbol{\sigma}_k = 0$ ensures the global convergence of $\mathbf{x}_k$ to $\mathbf{x}_d$ asymptotically. □

Remark 3. We can choose either learning control gain β_1 or feedback gain β_{fb} to be sufficiently high so as to achieve the fast convergence of the system states $\mathbf{x}$ to the desired value $\mathbf{x}_d$.

Remark 4. In the calculation of the bounding function ρ_j in equation (27), the derivative signal of the system state $\dot{z}_i$ is needed. It is then easy to observe that, if $\mathbf{x}_n$ is not included in $\mathbf{z}$, all the derivatives of $\dot{\mathbf{z}}$ are in fact measurable system states. In most motion control systems such as robotic manipulator, the dynamics can be expressed as

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2,$$

$$\dot{\mathbf{x}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}) + B(\mathbf{x}_1)\mathbf{u}.$$

Therefore, acceleration measurement is not required if using the proposed robust learning control scheme.

Remark 5. Control chattering of variable structure controller may deteriorate learning performance. This problem can be mitigated by inserting a small linear boundary layer around the origin of the error coordinates. The control system will guarantee uniform ultimate boundedness instead of asymptotic stability. This is due to the presence of the uncertainties $\mathbf{d}$ which requires essentially infinite gain across the switching surface $\boldsymbol{\sigma} = 0$.

3.5. *Robust ILC without resetting.* It is well known that iterative learning control schemes are very sensitive to the initial zeroing condition. Incomplete resetting or non-zero initial error, no matter how small is the initial error, may result in divergence of the learning control system. Therefore, from the practical point of view, it would be more important and interesting to investigate the condition under which the resetting requirement can be removed for the iterative type learning control. This is concluded in the following theorem associated with the assumption:

Assumption A4. For any element of the matrix B which is an explicit function of t, then it is also a periodic function with the period T_f, that is,

$$B(\mathbf{z}, \mathbf{p}, 0) = B(\mathbf{z}, \mathbf{p}, T_f). \tag{33}$$

Theorem 2 (*Robust ILC without resetting*). Assume that the alignment of the system state variables $\mathbf{x}_j(T_f) = \mathbf{x}_{j+1}(0)$ is ensured for any two consecutive trials, then learning control law (20) and variable structure control law (28) guarantee that the nonlinear system (1) tracks the given trajectory $\mathbf{x}_d$ of Category II asymptotically while all state variables are globally and uniformly bounded.

Proof. Under the condition $\mathbf{x}_j(T_f) = \mathbf{x}_{j+1}(0)$ and the property of the given trajectory which also ensures $\mathbf{x}_{j,d}(T_f) = \mathbf{x}_{j+1,d}(0)$, it follows that

$$\boldsymbol{\sigma}_j(T_f) = \boldsymbol{\sigma}_{j+1}(0) \tag{34}$$

is satisfied for all trials. From inequality (29) we know

$$\begin{aligned}\Delta V_j \le &-\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(T_f)B_j^{-1}(\mathbf{z}(T_f), T_f)\boldsymbol{\sigma}_j(T_f)\\ &+\beta_1\boldsymbol{\sigma}_j^{\mathrm{T}}(0)B_j^{-1}(\mathbf{z}(0), 0)\boldsymbol{\sigma}_j(0) - 2\int_0^{T_f}\beta_1\beta_{\mathrm{fb}}\|\boldsymbol{\sigma}_j\|^2\,\mathrm{d}\tau.\end{aligned}$$

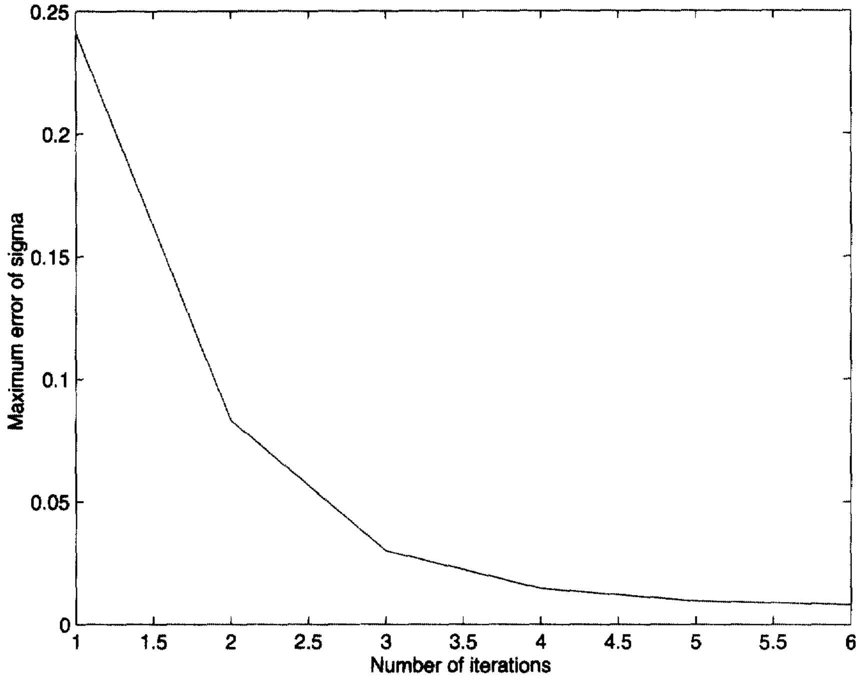

Fig. 1. The maximum tracking error profile of σ vs. iterations.

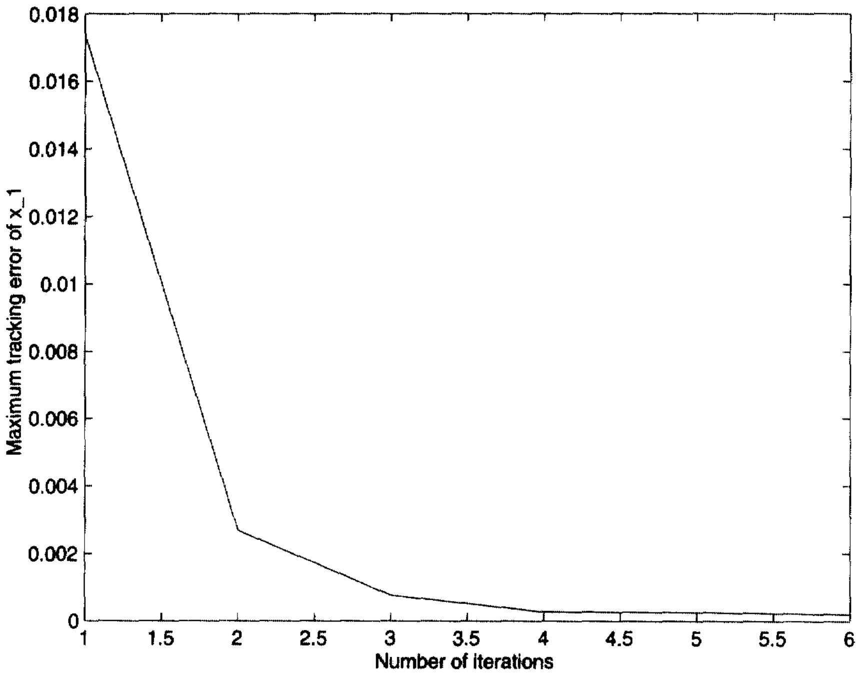

Fig. 2. The maximum tracking error profile of x_1 vs. iterations.

Again, taking summation of ΔV_j up to k and using conditions (33) and (34) yields

$$
\begin{aligned}
\sum_{j=0}^{k} \Delta V_j &= V_{k+1} - V_0 \\
&\le -\sum_{j=0}^{k} \beta_1 \sigma_j^{\mathrm{T}}(T_f) B^{-1}(\mathbf{z}_j(T_f), T_f)\sigma_j(T_f) \\
&\quad + \sum_{j=0}^{k} \beta_1 \sigma_j^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_j(0), 0)\sigma_j(0) \\
&\quad - \sum_{j=0}^{k} 2\beta_1\beta_{fb} \int_0^{T_f} \|\sigma_j\|^2 \,\mathrm{d}\tau \\
&= -\sum_{j=1}^{k+1} \beta_1 \sigma_j^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_j(0), 0)\sigma_j(0) \\
&\quad + \sum_{j=0}^{k} \beta_1 \sigma_j^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_j(0), 0)\sigma_j(0) \\
&\quad - \sum_{j=0}^{k} 2\beta_1\beta_{fb} \int_0^{T_f} \|\sigma_j\|^2 \,\mathrm{d}\tau
\end{aligned}
$$

$$= -\beta_l \boldsymbol{\sigma}_{k+1}^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_{k+1}(0), 0)\boldsymbol{\sigma}_{k+1}(0) + \beta_l \boldsymbol{\sigma}_0^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_0(0), 0)\boldsymbol{\sigma}_0(0) - \sum_{j=0}^{k} 2\beta_l \beta_{\mathrm{fb}} \int_0^{T_f} \|\boldsymbol{\sigma}_j\|^2 \,\mathrm{d}\tau. \quad (35)$$

From above formula we can derive

$$\sum_{j=0}^{k} 2\beta_l \beta_{\mathrm{fb}} \int_0^{T_f} \|\boldsymbol{\sigma}_j\|^2 \mathrm{d}\tau \le V_0 + \beta_l \boldsymbol{\sigma}_0^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_0(0), 0)\boldsymbol{\sigma}_0(0), \quad (36)$$

in which the right-hand side remains constant as k increases. Therefore, similar to Theorem 1, taking the limit of $k \to \infty$ leads to

$$\lim_{k\to\infty} \int_0^{T_f} \|\boldsymbol{\sigma}_k\|^2 \,\mathrm{d}\tau = 0.$$

Since the switching surface (6) is selected to be *Hurwitz*, $\boldsymbol{\sigma}_k = 0$ ensures the global convergence of $\mathbf{x}_k$ to $\mathbf{x}_d$ asymptotically.

Remark 6. The assumption imposed on the initial and terminal system states $\mathbf{x}_j(T_f) = \mathbf{x}_{j+1}(0)$ for any two consecutive trials is very reasonable for most motion control systems as the final position of the previous trial naturally becomes the initial position of the new trial.

Remark 7. If B matrix in equation (1) is autonomous, namely, no explicit time function in B, then the periodity Aassumption A4 is not necessary. It is obvious that, for most motion control systems including robotic dynamics, matrix B is autonomous. As a consequence, we can remove the resetting mechanism which is indispensable for conventional ILC schemes.

4. *Simulation example*

Consider the following second-order nonlinear dynamics

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \theta(t)x_1^2 x_2 + b(t)u + d(x_1, x_2, t). \quad (37)$$

The structured uncertainties are $\theta(t) = 9(1 + \cos(t))$ and $b(t) = 1 + e^{\sin(t)}$. $b(t)$ has a known lower bound $\lambda_{\min} = 1$ and $\dot{b}(t)$ has a known upper bound $\lambda_{\max} = 1$. The unstructured uncertainty is $d = 0.2\sin(x_1 x_2)$ with a known upper bound $d_{\max} = 0.3$.

The desired trajectory is $x_d(t) = \sin^3 t$ and the period is $[0, 2\pi]$. The switching surface is chosen to be

$$\sigma = (\dot{x}_d - x_2) + 10(x_d - x_1).$$

In the simulation the learning gain and the feedback gain are chosen to be $\beta_l = \beta_{\mathrm{fb}} = 10$. The sampling period is 1 ms. To reduce the chattering, the switching function $\mathrm{sgn}(\sigma_i)$ is replaced by the following continuous saturation function

$$\mathrm{sat}(\sigma) = \begin{cases} \mathrm{sgn}(\sigma) & \text{if } |\sigma| > 0.2, \\ \sigma/0.2 & \text{otherwise.} \end{cases} \quad (38)$$

Robust iterative learning is conducted without resetting. Figures 1 and 2 illustrate the tracking profiles of σ and the state x_1 vs. iterations respectively, which confirm the effectiveness of the proposed robust iterative learning control scheme.

5. *Conclusion*

A new control scheme—robust iterative learning control scheme is developed by integrating variable structure control and iterative learning control approaches. The proposed robust ILC system possesses both *learning* and robustness properties, thereby is able to handle quite general classes of nonlinear systems. The learning approach is used to attack the structured system uncertainties, whereas VSC approach is used to deal with non-structured uncertainties. Based on Lyapunov's direct method, in this paper we further investigate and clarify a number of important properties associated with the ILC scheme such as objective trajectory categories, the resetting condition, and the use of derivative signals.

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