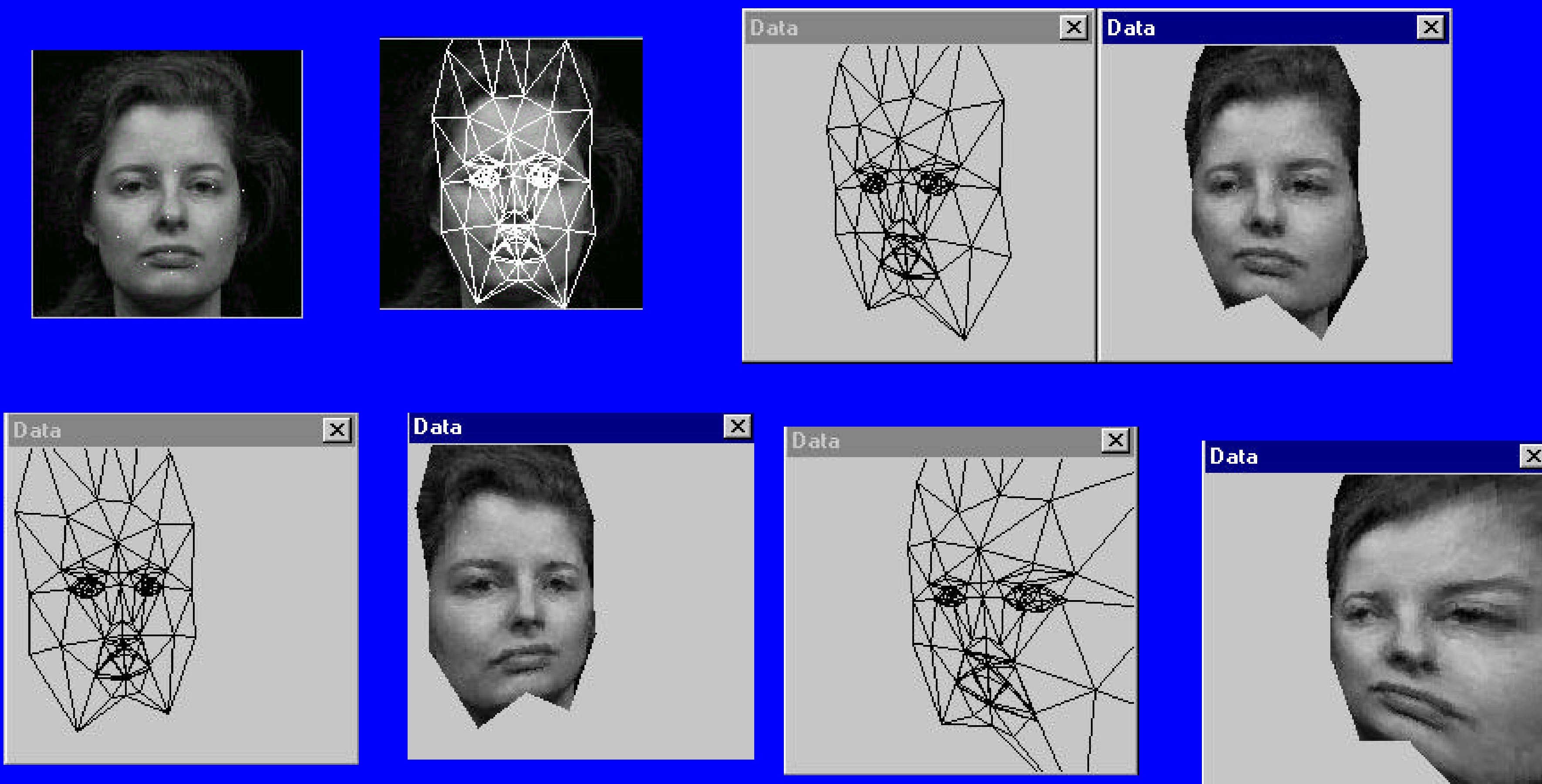
Lecture-2

Imaging Geometry

Transformations

- Translation
- Scaling
- Rotation
- Perspective
- Homogenous

Pose Estimation/Image Synthesis



Motion Estimation

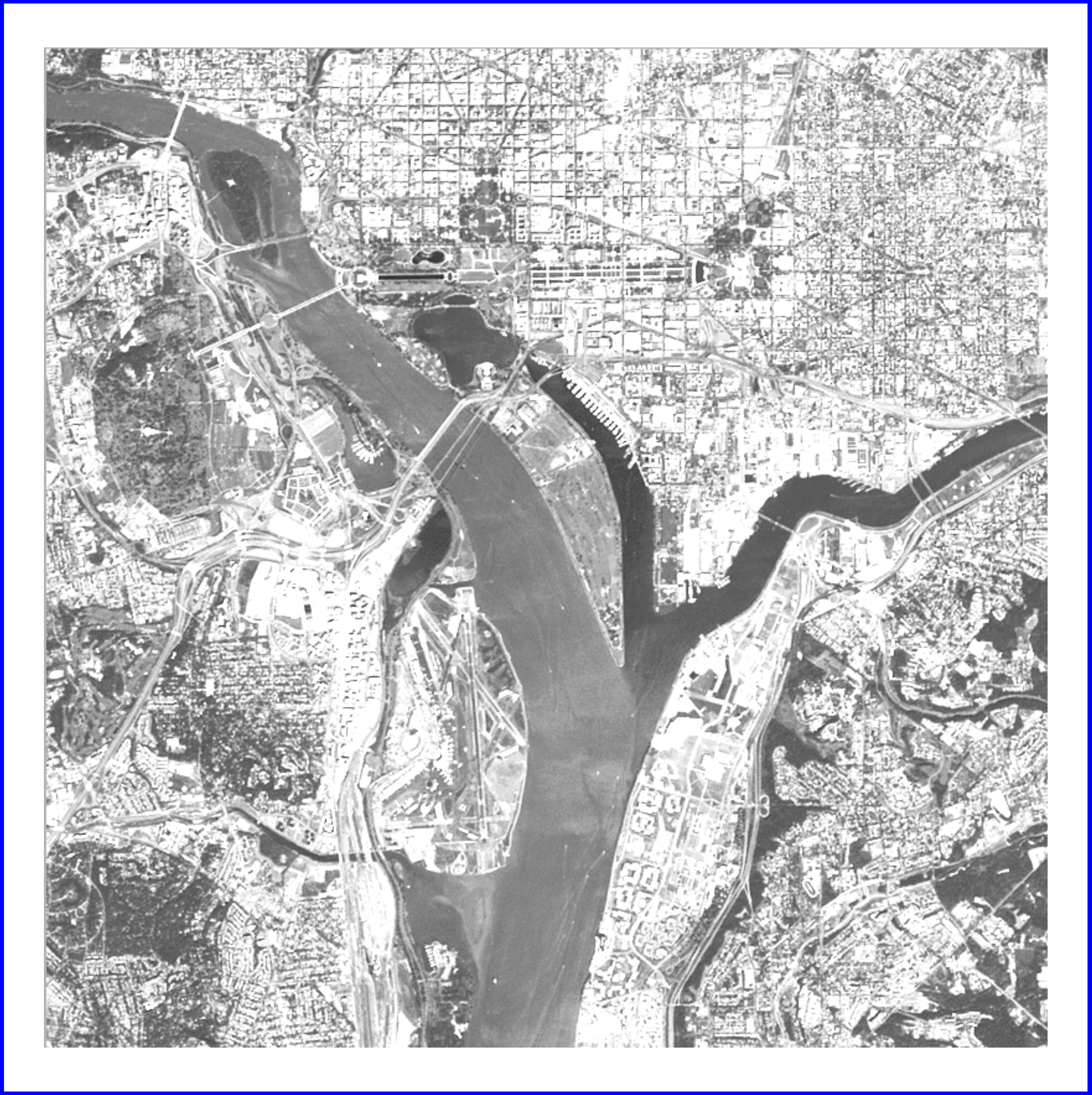
Motion Estimation

Object Recognition

- Robotics
- Image Registration

IRS-1C - Washington, DC

SPOT - Washington, DC



SPOT/IRS-1C Uncorrected

SPOT IRS-1C

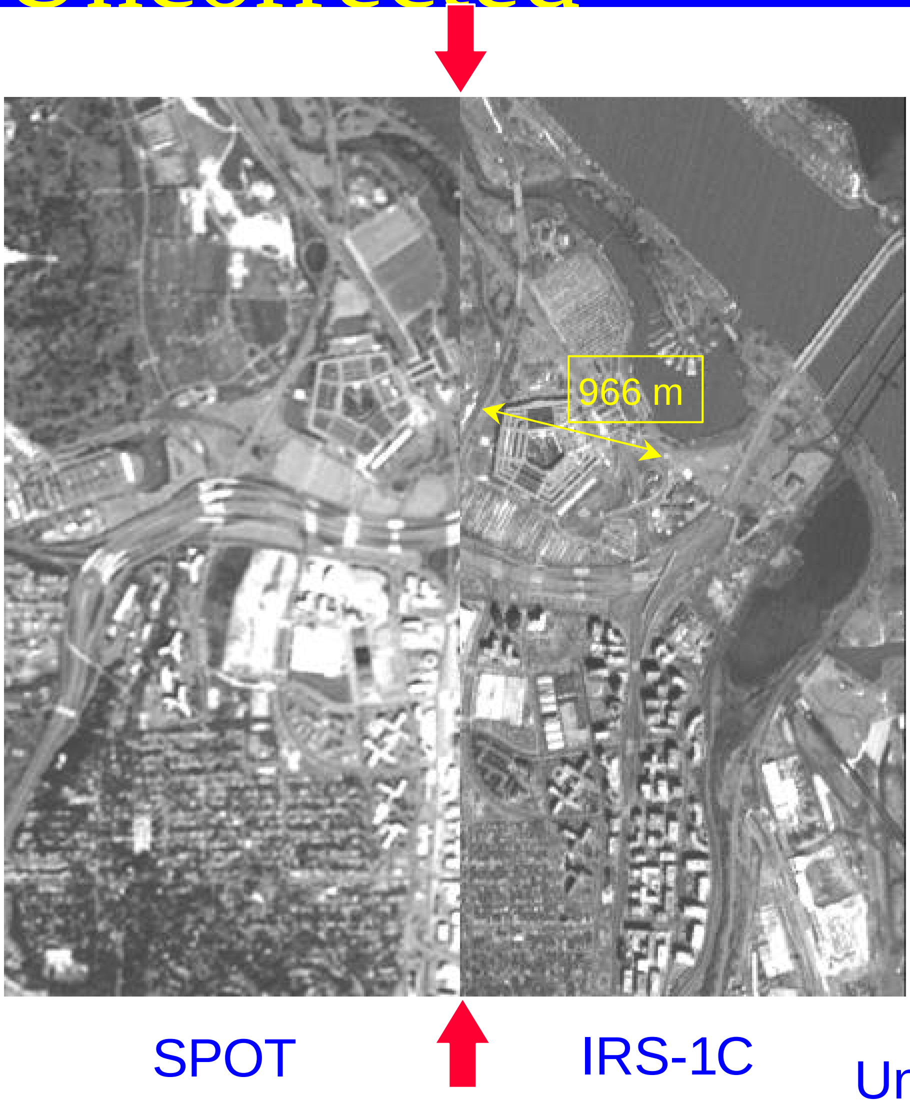
Uncorrected

SPOT/IRS-1C Uncorrected

SPOT IRS-1C

Uncorrected

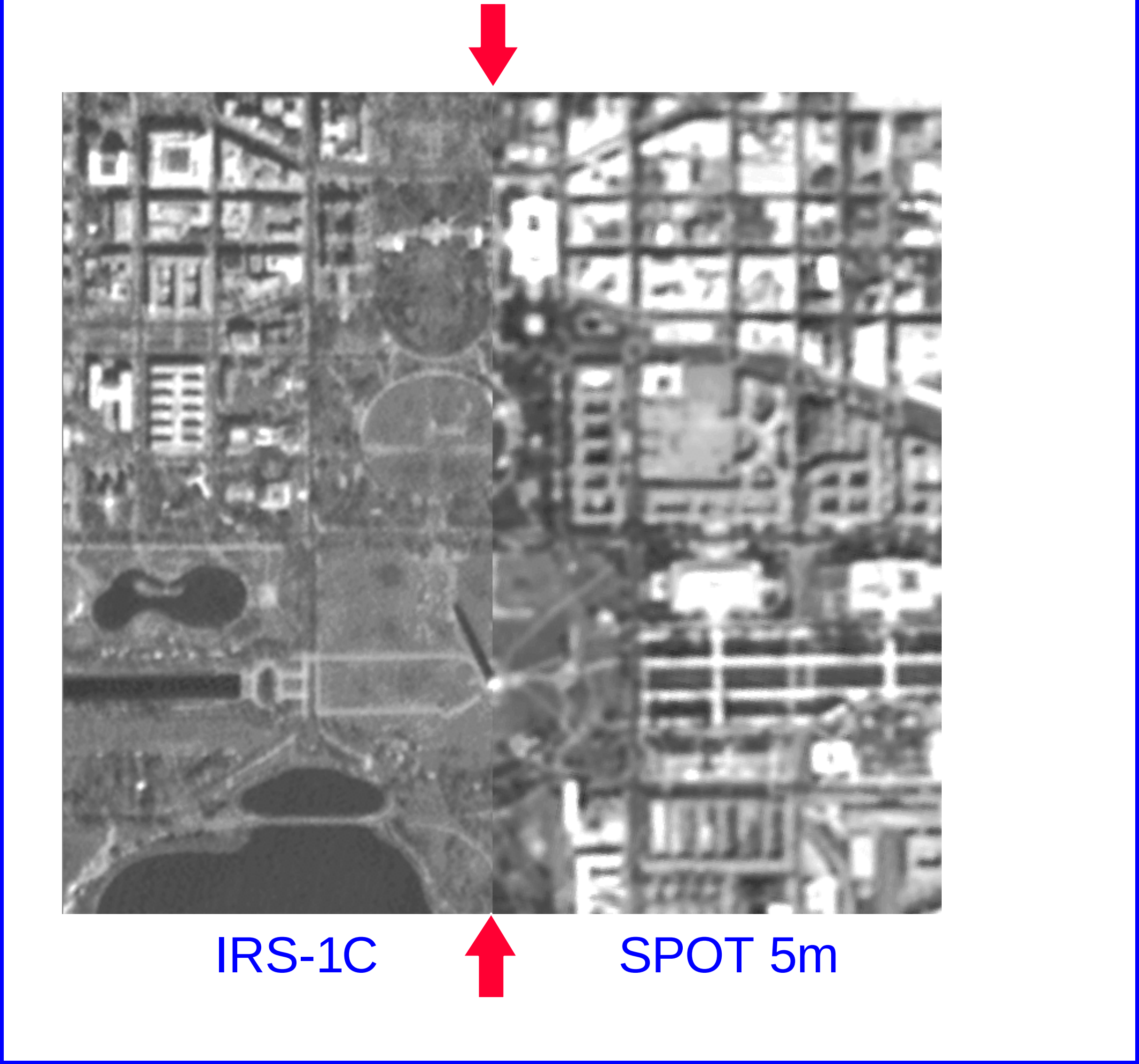
SPOT/IRS-1C Uncorrected



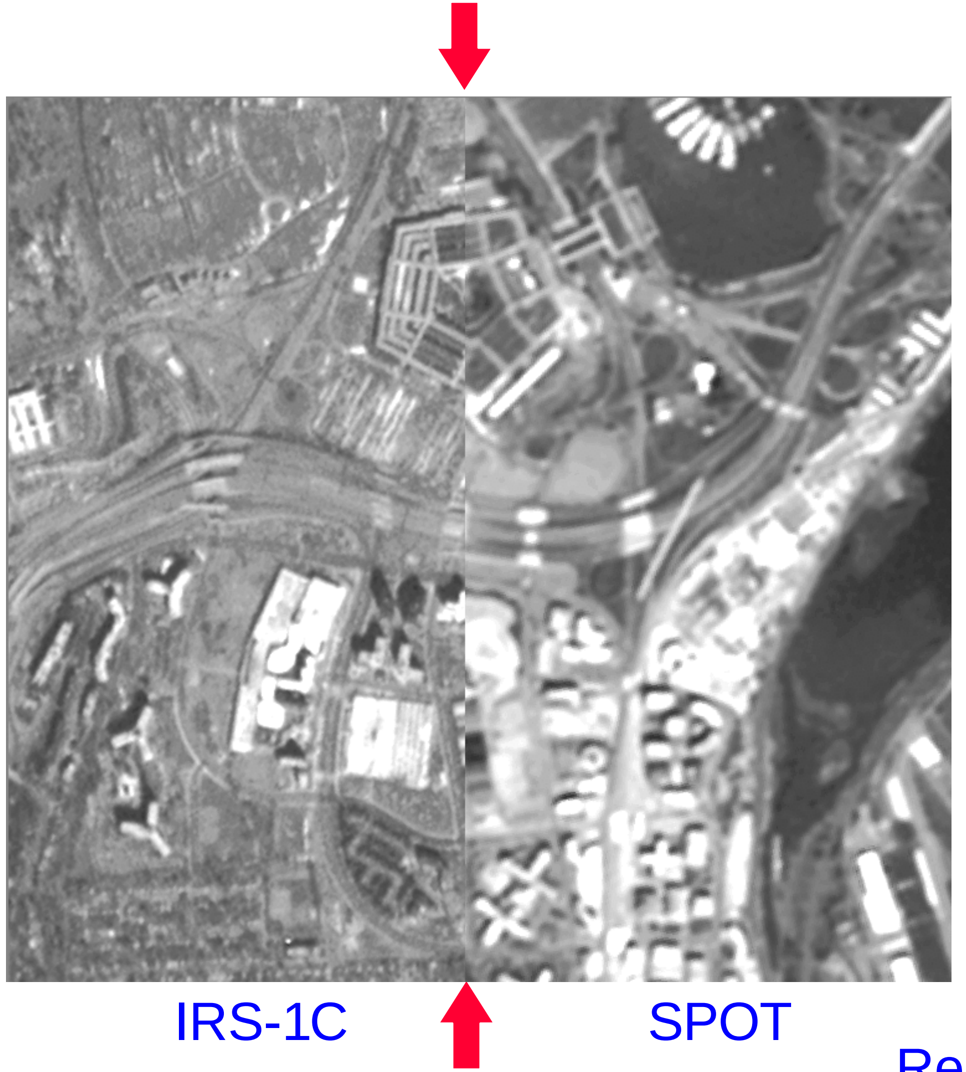
SPOT IRS-1C

Uncorrected

IRS-1C/SPOT Registered



Registered IRS-1C to SPOT



Translation

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = T \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Translatio n Matrix}$$

$$\mathrm{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -d_x \\ 0 & 1 & 0 & -d_y \\ 0 & 0 & 1 & -d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$TT^{-1} = T^{-1}T = I$$

$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -d_x \\ 0 & 1 & 0 & -d_y \\ 0 & 0 & 1 & -d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 \times S_x \\ Y_1 \times S_y \\ Z_1 \times S_z \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = S \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Scaling Matrix}$$

$$S^{-1} = \begin{bmatrix} 1/S_x & 0 & 0 & 0 \\ 0 & 1/S_y & 0 & 0 \\ 0 & 0 & 1/S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

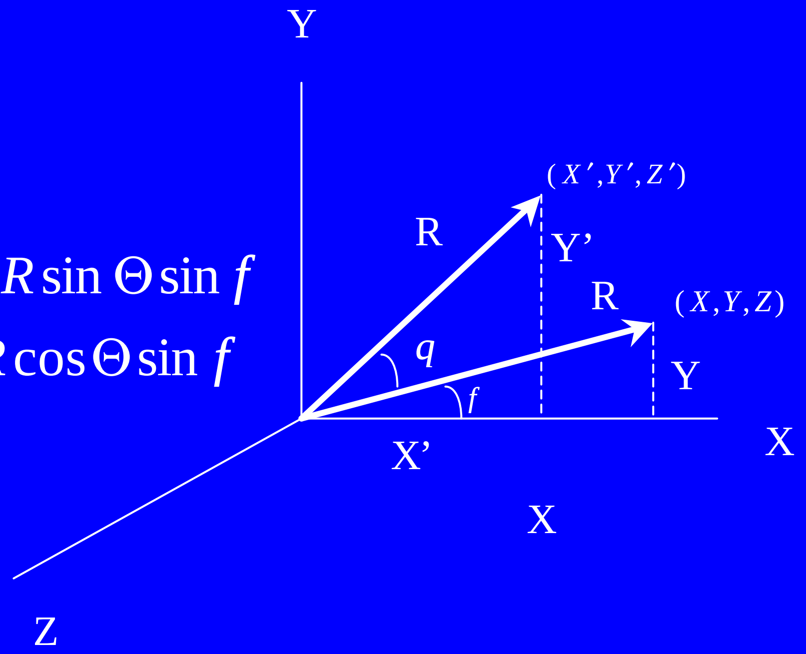
$$SS^{-1} = S^{-1}S = I$$

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/S_x & 0 & 0 & 0 \\ 0 & 1/S_y & 0 & 0 \\ 0 & 0 & 1/S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$X = R\cos f$$

$$Y = R\sin f$$



$$X' = R\cos(\Theta + f) = R\cos\Theta\cos f - R\sin\Theta\sin f$$

$$Y' = R\sin(\Theta + f) = R\sin\Theta\cos f + R\cos\Theta\sin f$$

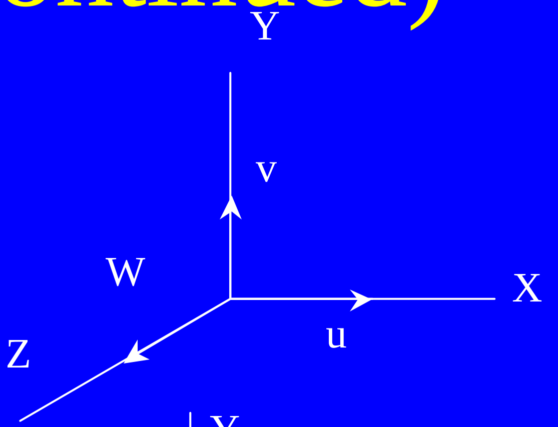
$$X' = X\cos\Theta - Y\sin\Theta$$

$$Y' = X\sin\Theta + Y\cos\Theta$$

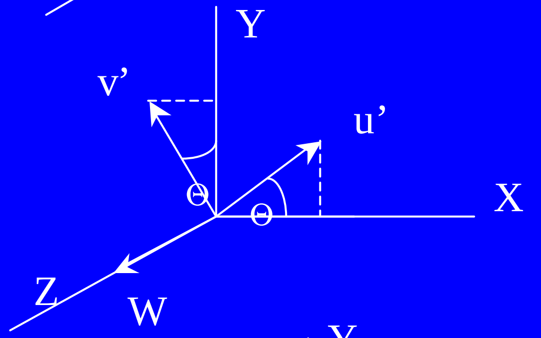
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Rotation (continued)

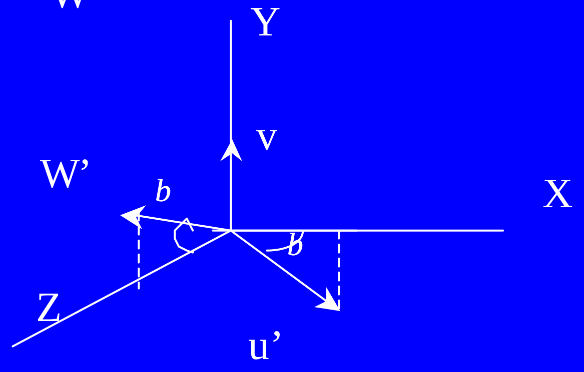
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_q^Z = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_{-b}^Y = \begin{bmatrix} \cos b & 0 & -\sin b \\ 0 & 1 & 0 \\ \sin b & 0 & \cos b \end{bmatrix}$$



$$(R_q^Z)^{-1} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R_q^Z)^{-1} = (R_q^Z)^T$$

$$(R_q^Z)(R_q^Z)^T = I$$

Rotation matrices are orthonormal matrices

$$r_i . r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Euler Angles

$$R = R_Z^a R_Y^b R_X^g = \begin{bmatrix} \cos a \cos b & \cos a \sin b \sin g - \sin a \cos g & \cos a \sin b \cos g + \sin a \sin g \\ \sin a \cos b & \sin a \sin b \sin g + \cos a \cos g & \sin a \sin b \cos g - \cos a \sin g \\ -\sin b & \cos b \sin g & \cos b \cos g \end{bmatrix}$$

if angles are small $\cos\Theta \approx 1 \quad \sin\Theta \approx \Theta$

$$R = \begin{bmatrix} 1 & -a & b \\ a & 1 & -g \\ -b & g & 1 \end{bmatrix}$$

Consider a point on 2 D plane undergoing non-affine transformation;

[X Y k] = [X/k Y/k 1] = [x y 1]

[3 2 k] = [3/k 2/k 1]

Start with k = 1

[3 2 1] represents the actual point (3,2)]

[3 2 0.5] = [6 4 1]

[3 2 0.1] = [30 20 1] actual point (30, 20)

[3 2 0.0001] = [30000 20000 1]

[3 2 0.0000001] = [30000000 20000000 1]

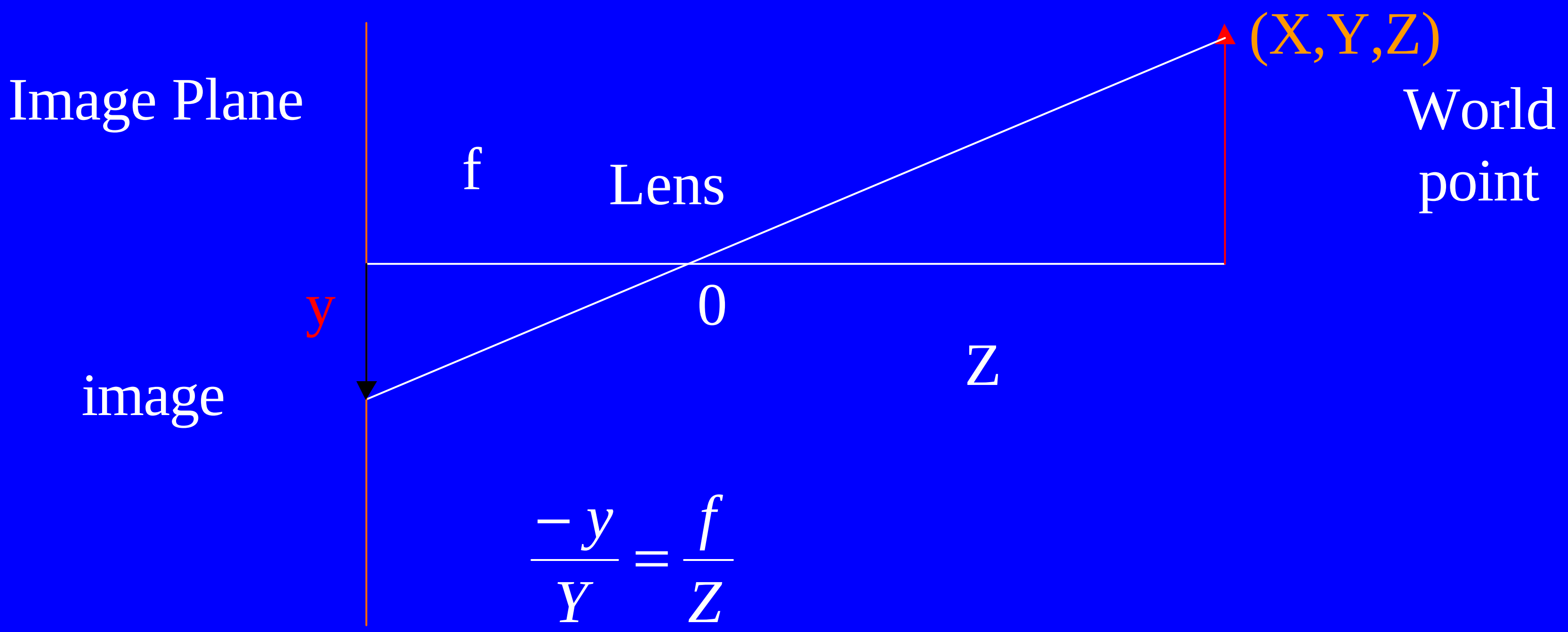
In the limit [3 2 0] = ??

[1 0 0] represents a point at infinity on the X axis

0 / 0 is taken as 1

[0 1 0] represents a point at infinity on the Y axis

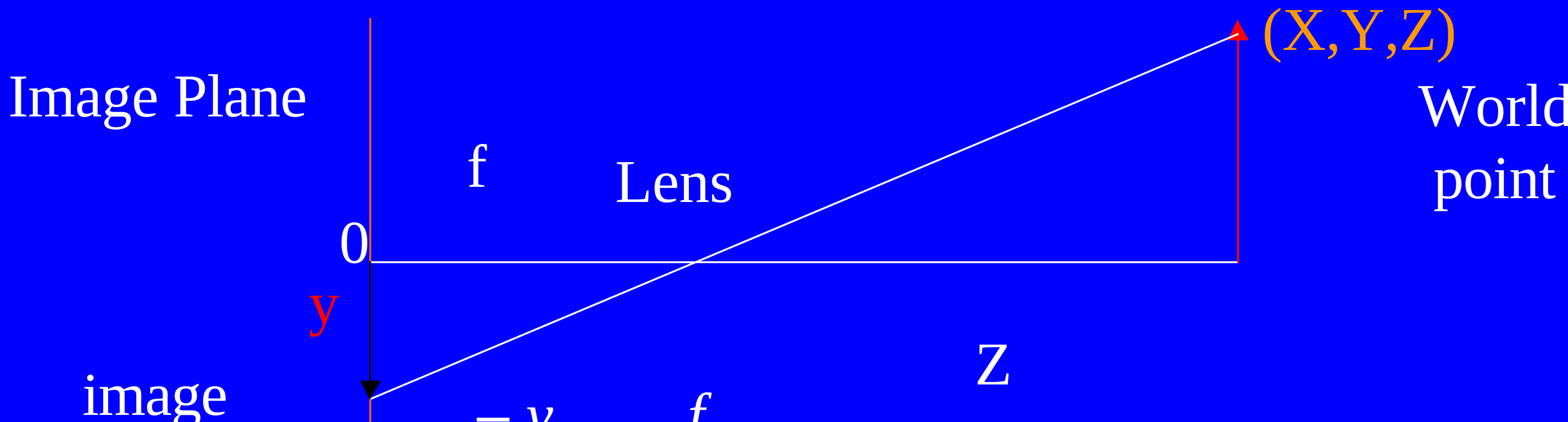
Perspective Projection (origin at the lens center)



$$\frac{-y}{Y} = \frac{f}{Z}$$

$$y = -\frac{fY}{Z} \qquad x = -\frac{fX}{Z}$$

Perspective Projection (origin at image center)



$$\frac{-y}{Y} = \frac{f}{Z-f}$$

$$y = -\frac{fY}{Z-f} \qquad x = -\frac{fX}{Z-f}$$

$$y = \frac{fY}{f-Z} \qquad x = \frac{fX}{f-Z}$$

Perspective

$$\begin{bmatrix} x \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} & & ? & & \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Image coordinates

World coordinates

$$(X, Y, Z) \rightarrow \;\; \rightarrow \rightarrow \qquad (kX, kY, kZ, k), \text{ Homogenous transformation}$$

$$(\mathrm{C}_{\mathrm{h1}}, C_{h2}, C_{h3}, C_{h4}) \rightarrow \qquad \left(\frac{\mathrm{C}_{\mathrm{h1}}}{C_{h4}}, \frac{C_{h2}}{C_{h4}}, \frac{C_{h3}}{C_{h4}}\right), \text{ Inverse homogenous}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{f} & 1 \end{bmatrix}$$

Perspective

$$\begin{bmatrix} C_{h1} \\ C_{h2} \\ C_{h3} \\ C_{h4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{f} & 1 \end{bmatrix} \begin{bmatrix} kX \\ kY \\ kZ \\ k \end{bmatrix}$$

$$\begin{bmatrix} C_{h1} \\ C_{h2} \\ C_{h3} \\ C_{h4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{f} & 1 \end{bmatrix} \begin{bmatrix} kX \\ kY \\ kZ \\ k \end{bmatrix} = \begin{bmatrix} kX \\ kY \\ kZ \\ k - \frac{kZ}{f} \end{bmatrix}$$

$$x = \frac{C_{h1}}{C_{h4}} = \frac{kX}{k - \frac{kZ}{f}} = \frac{fX}{f - Z}$$

$$y = \frac{C_{h2}}{C_{h4}} = \frac{kY}{k - \frac{kZ}{f}} = \frac{fY}{f - Z}$$

Camera Model

- Camera is at the origin of the world coordinates first
- Then translated (G),
- then rotated around Z axis in counter clockwise direction,
- then rotated again around X in counter clockwise direction, and
- then translated by C.

$$C_h = PCR_{-f}^{X} R_{-q}^{Z} GW_h$$

Camera Model

$$C_h = PCR^X_{-f} R^Z_{-q} G W_h$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{f} & 1 \end{bmatrix}, R^Z_{-q} = \begin{bmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^X_{-f} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos f & \sin f & 0 \\ 0 & -\sin f & \cos f & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & -X_0 \\ 0 & 1 & 0 & -Y_0 \\ 0 & 0 & 1 & -Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & -r_1 \\ 0 & 1 & 0 & -r_2 \\ 0 & 0 & 1 & -R_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera Model

$$C_h = PCR^{X}_{-f}R^{Z}_{-q}GW_h$$

$$x = f\,\frac{(X - X_0)\cos q + (Y - Y_0)\sin q - r_1}{-(X - X_0)\sin q \sin f + (Y - Y_0)\cos q \sin f - (Z - Z_0)\cos f + r_3 + f}$$

$$y = f\,\frac{(X - X_0)\sin q \cos f + (Y - Y_0)\cos q \cos f + (Z - Z_0)\sin f - r_2}{-(X - X_0)\sin q \sin f + (Y - Y_0)\cos q \sin f - (Z - Z_0)\cos f + r_3 + f}$$

Camera Model

$$C_h = PCR_{-f}^{X} R_{-q}^{Z} G W_h$$

$$C_h = A W_h$$

$$\begin{bmatrix} Ch_1 \\ Ch_2 \\ Ch_3 \\ Ch_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x = \frac{C_{h1}}{C_{h4}}$$

$$y = \frac{C_{h2}}{C_{h4}}$$

$$Ch_1 = a_{11}X + a_{12}Y + a_{13}Z + a_{14} = Ch_4 x$$

$$Ch_2 = a_{21}X + a_{22}Y + a_{23}Z + a_{24} = Ch_4 y$$

$$Ch_4 = a_{41}X + a_{42}Y + a_{43}Z + a_{44}$$

Camera Model

$$Ch_1 = a_{11}X + a_{12}Y + a_{13}Z + a_{14} = Ch_4 x$$

$$Ch_2 = a_{21}X + a_{22}Y + a_{23}Z + a_{24} = Ch_4 y$$

$$Ch_4 = a_{41}X + a_{42}Y + a_{43}Z + a_{44}$$

$$a_{11}X + a_{12}Y + a_{13}Z + a_{14} - a_{41}Xx - a_{42}Yx - a_{43}Zx - a_{44}x = 0$$

$$a_{21}X + a_{22}Y + a_{23}Z + a_{24} - a_{41}Xy - a_{42}Yy - a_{43}Zy - a_{44}y = 0$$

Camera Model

$$a_{11}X + a_{12}Y + a_{13}Z + a_{14} - a_{41}Xx - a_{42}Yx - a_{43}Zx - a_{44}x = 0$$

$$a_{21}X + a_{22}Y + a_{23}Z + a_{24} - a_{41}Xy - a_{42}Yy - a_{43}Zy - a_{44}y = 0$$

One point

$$a_{11}X_1 + a_{12}Y_1 + a_{13}Z_1 + a_{14} - a_{41}X_1x_1 - a_{42}Y_1x_1 - a_{43}Z_1x_1 - a_{44}x_1 = 0$$

$$a_{11}X_2 + a_{12}Y_2 + a_{13}Z_2 + a_{14} - a_{41}X_2x_2 - a_{42}Y_2x_2 - a_{43}Z_2x_2 - a_{44}x_2 = 0$$

$$\vdots$$

$$a_{11}X_n + a_{12}Y_n + a_{13}Z_n + a_{14} - a_{41}X_nx_n - a_{42}Y_nx_n - a_{43}Z_nx_n - a_{44}x_n = 0$$

$$a_{21}X_1 + a_{22}Y_1 + a_{23}Z_1 + a_{24} - a_{41}X_1y_1 - a_{42}Y_1y_1 - a_{43}Z_1y_1 - a_{44}y_1 = 0$$

$$a_{21}X_2 + a_{22}Y_2 + a_{23}Z_2 + a_{24} - a_{41}X_2y_2 - a_{42}Y_2y_2 - a_{43}Z_2y_2 - a_{44}y_2 = 0$$

$$\vdots$$

$$a_{21}X_n + a_{22}Y_n + a_{23}Z_n + a_{24} - a_{41}X_ny_n - a_{42}Y_ny_n - a_{43}Z_ny_n - a_{44}y_n = 0$$

n points
2n equations,
12 unknowns

Camera Model

$$
\begin{bmatrix}
X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 & -x_1Z_1 & -x_1 \\
X_2 & Y_2 & Z_2 & 1 & 0 & 0 & 0 & 0 & -x_2X_2 & -x_2X_2 & -x_2X_2 & -x_2 \\
 & & & & & & \vdots & & & & & \\
X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -x_nX_n & -x_nY_n & -x_nZ_n & -x_n \\
0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -y_1X_1 & -y_1X_1 & -y_1Z_1 & -y_1 \\
0 & 0 & 0 & 0 & X_2 & Y_2 & Z_2 & 1 & -y_2X_2 & -y_2X_2 & -y_2Z_2 & -y_2 \\
 & & & & & & \vdots & & & & & \\
0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -y_nX_n & -y_nX_n & -y_nX_n & -y_n
\end{bmatrix}
\begin{bmatrix}
a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{41} \\ a_{42} \\ a_{43} \\ a_{44}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

$$CP = 0$$

Camera Model

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 & -x_1Z_1 \\ X_2 & Y_2 & Z_2 & 1 & 0 & 0 & 0 & 0 & -x_2X_2 & -x_2X_2 & -x_2X_2 \\ & & & & & & \vdots & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -x_nX_n & -x_nY_n & -x_nZ_n \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -y_1X_1 & -y_1X_1 & -y_1Z_1 \\ 0 & 0 & 0 & 0 & X_2 & Y_2 & Z_2 & 1 & -y_2X_2 & -y_2X_2 & -y_2Z_2 \\ & & & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -y_nX_n & -y_nX_n & -y_nX_n \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{41} \\ a_{42} \\ a_{43} \\ \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ \\ x_n \\ y_1 \\ y_2 \\ \\ \\ \\ y_n \end{bmatrix}$$

$$DQ = R$$

$$D^T DQ = D^T R$$

$$Q = (D^T D)^{-1} D^T R$$

Camera Model Revisited

$$P_c = RTP_w = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & -T_y \\ 0 & 0 & 1 & -T_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$P_c = R(P_w - T) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{bmatrix} X - T_x \\ Y - T_y \\ Z - T_z \end{bmatrix}$$

$$P_c = R(P_w - T) = \begin{pmatrix} R_1^T \\ R_2^T \\ R_3^T \end{pmatrix} \left[P_w - T \right]$$

$$x_c = f \frac{R_1^T (P_w - T)}{R_3^T (P_w - T)}$$

$$y_c = f \frac{R_2^T (P_w - T)}{R_3^T (P_w - T)}$$

Perspective projection

Camera Model Revisited

$$x = -(x_{im} - o_x)s_x$$

$$y = -(y_{im} - o_y)s_y$$

(x_{im}, y_{im}) image coordinates

(x, y) camera coordinates

(o_x, o_y) image center (in pixels)

(s_x, s_y) effective size of pixels (in millimeters) in the horizontal and vertical directions.

$$-(x_{im} - o_x)s_x = f\,\frac{R_1^T(P_w - T)}{R_3^T(P_w - T)}$$

$$-(y_{im} - o_y)s_y = f\,\frac{R_2^T(P_w - T)}{R_3^T(P_w - T)}$$

Camera Model Revisited

$$M_{\text{int}} = \begin{bmatrix} -\frac{f}{s_x} & 0 & o_x \\ 0 & -\frac{f}{s_x} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & -R_1^T T \\ r_{21} & r_{22} & r_{23} & -R_2^T T \\ r_{31} & r_{32} & r_{33} & -R_3^T T \end{bmatrix}$$

$$M = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & fR_1^T T \\ -fr_{21} & -fr_{22} & -fr_{23} & fR_2^T T \\ r_{31} & r_{32} & r_{33} & -R_3^T T \end{bmatrix}$$

$$\begin{bmatrix} Ch_1 \\ Ch_2 \\ Ch_3 \end{bmatrix} = M_{\text{int}} M_{ext} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

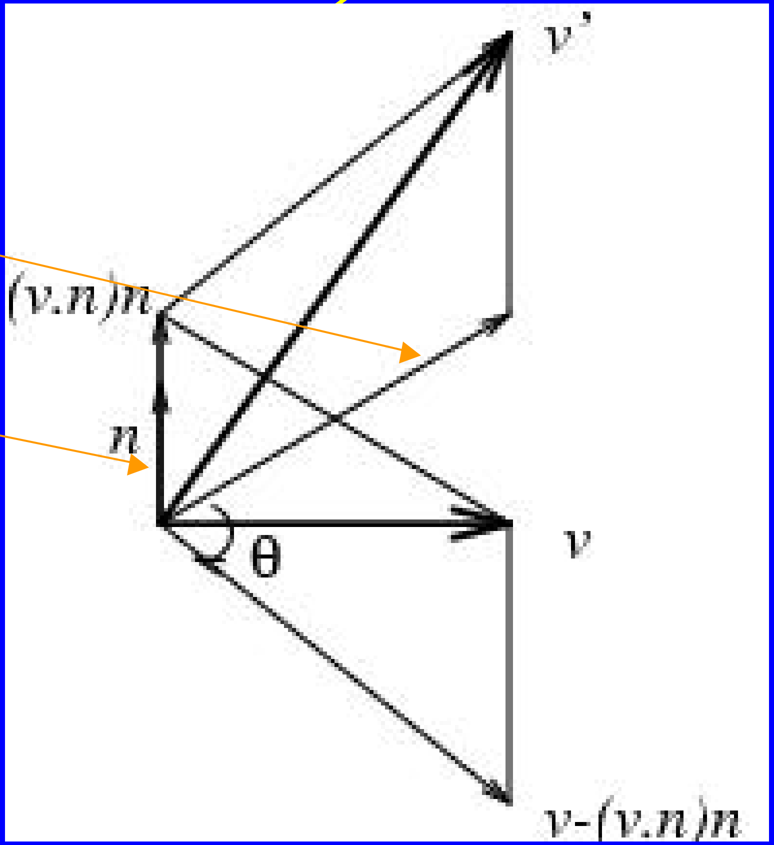
Rotation Around an Arbitrary Axis (Rodriguez's Formula)

$$V = (V.n)n + (V - (V.n)n)$$

$$V'_{\perp} = \cos q\,(V - (V.n)n) + \sin q\,(n \times (V - (V.n)n)$$

$$V'_{\mathrm{II}} = (V.n)n$$

$$V' = V'_{\perp} + V'_{\mathrm{II}}$$



$$V' = \cos q\; V + \sin q\; n \times V + (1 - \cos q)(V.n)n$$

$$V' = V + \sin q\; n \times V + (1 - \cos q)\, n \times (n \times V)$$

$$n \times (n \times V) = (V.n)n - V$$

$$n \times (n \times V) + V = (V.n)n$$

Rotation Around an Arbitrary Axis (Rodriguez's Formula)

$$V' = V + \sin\theta\; n \times V + (1-\cos\theta)\; n \times (n \times V)$$

$$V' = R(n,\theta)V$$

$$R(n,\theta) = I + \sin\theta\; X(n) + (1-\cos\theta) X^2(n)$$

$$X(n) = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

Rotation Around an Arbitrary Axis (Rodriguez's Formula)

$$r = \|r\| \frac{r}{\|r|} = qn$$

$$R(n,q) = I + \sin q \; X(n) + (1-\cos q) X^2(n)$$

$$R(r,q) = I + \sin q \; X(\frac{r}{\|r\|}) + (1-\cos q) X^2(\frac{r}{\|r\|})$$

$$X(\frac{r}{\|r\|}) = \begin{bmatrix} 0 & -\frac{r_z}{\|r\|} & \frac{r_y}{\|r\|} \\ \frac{r_z}{\|r\|} & 0 & -\frac{r_x}{\|r\|} \\ -\frac{r_y}{\|r\|} & \frac{r_x}{\|r\|} & 0 \end{bmatrix}$$

$$\frac{X(r)}{\|r\|} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$R(r,q) = I + \sin q \frac{X(r)}{\|r\|} + (1-\cos q) \frac{X^2(r)}{\|r\|^2}$$