COT 4500 Quiz #5 Solutions
Date: 3/23/2012

1) (25 pts) Find the solution to the following differential equation with the initial condition $y(\pi)=3\pi^2$:

$$y - \frac{x}{2}y' + x^3\cos(2x) = 0$$

Solution

$y - \frac{x}{2}y' + x^3\cos(2x) = 0$, multiply through by $-\frac{2}{x}$

$-\frac{2}{x}y + y' - 2x^2\cos(2x) = 0$

$y' - \frac{2}{x}y = 2x^2\cos(2x)$, use integrating factor

Integrating factor = $e^{\int \frac{-2}{x}dx} = e^{-2\ln|x|} = x^{-2}$.

$\frac{y\prime}{x^2} - \frac{2y}{x^3} = 2\cos(2x)$, integrate both sides

$\frac{y}{x^2} = \sin(2x) + C$

$y = x^2\sin(2x) + Cx^2$

Plugging in x = π, y = $3\pi^2$, we get:

$3\pi^2 = C\pi^2$, so C = 3.

The final solution is $\boldsymbol{y = x^2\sin(2x) + 3x^2}$.

2) (25 pts) Use Euler's method to approximate the solution to the following differential equation with the initial condition y(4) = 10 in the interval $4 \le t \le 8$ and n = 5:

$$y' = \frac{\sqrt{t}}{y}$$

Please show your approximation for y for each of the five values of t at which we evaluate y during the algorithm.

Solution

Using the following algorithm shown in class:

```
double w[N+1];

double a = 4, b = 8;
w[0] = 10;
double h = (b - a)/N;
int i;

for (i=1; i<=N; i++) {
    w[i] = w[i-1] + h*f(a+(i-1)*h, w[i-1]);
}
```

we get the following values for the approximation of the function:

i	t	w
0	4	10
1	4.8	10.16
2	5.6	10.332511
3	6.4	10.515733
4	7.2	10.708193
5	8.0	10.908659

3) (25 pts) Use the Runge-Kutta method of order 4 to approximate the solution to the differential equation from question number 2 in the same interval, but use n = 2. To get full credit, please list out the values of k_1, k_2, k_3, and k_4 for both iterations of the algorithm as well as w_1 and w_2.

Solution

Using the algorithm shown in class:

```
double w[N+1];

double a = 4, b = 8;
w[0] = 10;

double h = (b - a)/N;
int i;

for (i=0; i<N; i++) {
  double ti = a+i*h;
  double tnext = a+(i+1)*h;

  double k1 = h*f(ti, w[i]);
  double k2 = h*f(ti+h/2,w[i]+k1/2);
  double k3 = h*f(ti+h/2,w[i]+k2/2);
  double k4 = h*f(tnext, w[i]+k3);
  w[i+1] = w[i] + (k1+2*k2+2*k3+k4)/6;

  printf("%lf %lf %lf %lf %lf %lf\n",a+i*h,k1,k2,k3,k4,w[i+1]);
}
```

the output is as follows:

i	t	k_1	k_2	k_3	k_4	w
1	6	.4	.438445	.437620	.469358	10.436915
2	8	.469390	.495849	.495235	.517451	10.931749

4) (25 pts) Find the solution to the following differential equation with the initial conditions $y(0)=4$, $y(1)=6e^5$.

$$y'' - 10y' + 25y = 0$$

Solution

The characteristic equation is:

$m^2 - 10m + 25 = 0$
$(m - 5)^2 = 0$

Thus, $m = 5$ is a repeated root. The general solution is:

$y = (c_0 + c_1x)e^{5x}$.

Now, plug in the initial conditions to get the following two equations:

$4 = c_0$
$6e^5 = (c_0 + c_1)e^5$

Divide the second equation by e^5 to yield:

$6 = c_0 + c_1$

Since $c_0 = 4$, it follows that $c_1 = 2$. The final equation is:

$y = (4 + 2x)e^{5x} = 2(x + 2)e^{5x}$.