Summations

To find the total number of operations in a program involving simple variables, arrays or linked lists, and being computed within a set of loops, find the number of times the loop is getting executed and the number of operations within the loop. The number of operations can be found with the help of summations. There can be a single loop or nested loops. Start with the innermost loop and work outwards.

Let us consider various cases below:

Case 1: Summation of constant values

Consider a single loop with a single operation. The summation with the loop variable moving from 1 to n would be given by

$$\sum_{j=1}^{n} 1$$

= 1 + 1 + 1 + … + 1 (n times)
= n

Consider another summation with 4 operations:

$$\sum_{j=1}^{n} 4$$

The constant term can be taken out of the summation.

$$4\sum_{j=1}^{n} 1$$

= 4n
Here is another example, where the constant term n can be taken out of the summation;

$$\sum_{j=1}^{n} n$$

$$= n\sum_{j=1}^{n} 1 \quad = n.n \quad = n^2$$

Take care if the loop starts from 0 instead of starting from 1. You would need to add one more term to the summation.

$$\sum_{j=0}^{n} 4 \quad = 4(n-0+1) \quad = 4(n+1)$$

Now let us consider a summation, where the lower limit is different from 1

$$\sum_{j=5}^{10} 4 \quad = 4(10-5+1) \quad = 4(6) = 24$$

Case 2: Summation involving the loop variable

When the number of operations depends on the loop variable, then the summation can take the following form

$$\sum_{j=1}^{n} j \quad = 1 + 2 + 3 + \ldots\ldots. + n$$

This is a well known mathematical series whose sum is expressed through the formulae:

$$n(n+1)/2$$

Case 3: Summation involving the loop variable & a constant

Now consider summation of a constant and a loop variable term

$$\sum_{j=1}^{10} 4 + j$$

This summation can be split into two summations and evaluated separately

$$\sum_{j=1}^{10} 4 + \sum_{j=1}^{10} j$$

= 4 (10) + (10) (11)/2
= 40 + 55

What happens when the lower limit is not 1? When it is 0, as in the following example

$$\sum_{j=0}^{n} j$$

= 10 (11)/ 2

which is same as the summation with lower limit as 1, since the first term is zero and effectively the summation is being carried out for j values from 1 to 10.

When the lower limit is different from 1 or 0, such as in the following summation, we split the summation in two parts

$$\sum_{j=5}^{10} j$$

$$= \sum_{j=1}^{10} j - \sum_{j=1}^{4} j$$

= 10 (11) / 2 – 4 (5) / 2

= 45

Case 4: Summation involving a different variable

Let us now consider the summation where the loop variable is different from the variable inside the summation, as in

$$\sum_{j=1}^{10} k$$

Since the variable k is not the loop variable j, it can be treated as a constant and can be moved out of the summation, leaving a 1 inside.

= 10 k

Case 5: Double Summations

When you have two nested loops, you would get a summation inside another summation. These are solved by working out the summations right to left. Consider for example,

$$\sum_{k=1}^{10} \sum_{j=1}^{6} j$$

The summation on the right is a case of a loop variable, and this allows us to reduce the problem to

$$\sum_{k=1}^{10} 6(7)/2$$

$$= \sum_{k=1}^{10} 21$$

Now, it is a simple case of summation over a constant, and the final result is
= 21 (10)
= 210