Finding the Difference Equation for Approximating Continuous-time Systems

Given the first order system $\dfrac{dy}{dt} = f(y,u)$,

forward (explicit) Euler: $y_A(n+1) = y_A(n) + Tf[y_A(n), u(n)]$

backward (implicit) Euler: $y_A(n+1) = y_A(n) + Tf[y_A(n+1), u(n+1)]$

trapezoidal: $y_A(n+1) = y_A(n) + \dfrac{T}{2}\left\{ f[y_A(n), u(n)] + f[y_A(n+1), u(n+1)] \right\}$

example: $\dfrac{dP}{dt} = f(P,u) = cP(P_m - P) + u$ where $u = u(t)$

forward (explicit) Euler:

$$P_A(n+1) = P_A(n) + Tf[P_A(n), u(n)]$$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + T\left[cP_A(n)\left\{ P_m - P_A(n) \right\} + u(n) \right]$$

$$\Rightarrow \quad P_A(n+1) = \left[1 + cT\left\{ P_m - P_A(n) \right\} \right] P_A(n) + Tu(n)$$

backward (implicit) Euler:

$$P_A(n+1) = P_A(n) + Tf[P_A(n+1), u(n+1)]$$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + T\left[cP_A(n+1)\left\{ P_m - P_A(n+1) \right\} + u(n+1) \right]$$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + cTP_m P_A(n+1) - cTP_A{}^2(n+1) + Tu(n+1)$$

$$\Rightarrow \quad cTP_A{}^2(n+1) - cTP_m P_A(n+1) + P_A(n+1) = P_A(n) + Tu(n+1)$$

$$\Rightarrow \quad cTP_A{}^2(n+1) + (1 - cTP_m)P_A(n+1) - P_A(n) - Tu(n+1) = 0$$

trapezoidal:

$$P_A(n+1) = P_A(n) + \frac{T}{2}\left\{ f[P_A(n), u(n)] + f[P_A(n+1), u(n+1)] \right\}$$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + \frac{T}{2}\left\{ \left[cP_A(n)\left\{ P_m - P_A(n) \right\} + u(n) \right] + \left[cP_A(n+1)\left\{ P_m - P_A(n+1) \right\} + u(n+1) \right] \right\}$$

example: $$\frac{dx}{dt} = f(x,y,u) = ax + by + u$$

$$\frac{dy}{dt} = g(x,y,u) = cxy$$

forward Euler: $$x_A(n+1) = x_A(n) + Tf[x_A(n), y_A(n), u(n)]$$

$$\Rightarrow \quad x_A(n+1) = x_A(n) + T\left[ax_A(n) + by_A(n) + u(n)\right]$$

$$\Rightarrow \quad x_A(n+1) = (1+aT)x_A(n) + bTy_A(n) + Tu(n)$$

$$y_A(n+1) = y_A(n) + Tg[x_A(n), y_A(n), u(n)]$$

$$\Rightarrow \quad y_A(n+1) = y_A(n) + T\left[cx_A(n)y_A(n)\right]$$

$$\Rightarrow \quad y_A(n+1) = \left[1 + cTx_A(n)\right]y_A(n)$$

example: $$\frac{dP_1}{dt} = cP(P_m - P) - \alpha P_2$$

$$\frac{dP_2}{dt} = \beta P_2 - \lambda P_1$$

$$f_1(P_1, P_2) = \frac{dP_1}{dt} = cP(P_m - P) - \alpha P_2$$

$$f_2(P_1, P_2) = \frac{dP_2}{dt} = \beta P_2 - \lambda P_1$$

trapezoidal: $$P_{1,A}(n+1) = P_{1,A}(n) + \frac{T}{2}\left\{f_1\left[P_{1,A}(n), P_{2,A}(n)\right] + f_1\left[P_{1,A}(n+1), P_{2,A}(n+1)\right]\right\}$$

$$\Rightarrow \quad P_{1,A}(n+1) = P_{1,A}(n) + \frac{T}{2}\left\{\left[cP_{1,A}(n)\left\{P_m - P_{1,A}(n)\right\} - \alpha P_{2,A}(n)\right] + \left[cP_{1,A}(n+1)\left\{P_m - P_{1,A}(n+1)\right\} - \alpha P_{2,A}(n+1)\right]\right\}$$

$$P_{2,A}(n+1) = P_{2,A}(n) + \frac{T}{2}\left\{f_2\left[P_{1,A}(n), P_{2,A}(n)\right] + f_2\left[P_{1,A}(n+1), P_{2,A}(n+1)\right]\right\}$$

$$\Rightarrow \quad P_{2,A}(n+1) = P_{2,A}(n) + \frac{T}{2}\left\{\left[\beta P_{2,A}(n) - \lambda P_{1,A}(n)\right] + \left[\beta P_{2,A}(n+1) - \lambda P_{1,A}(n+1)\right]\right\}$$