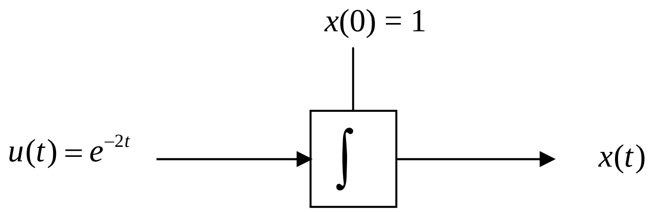
EEL 4890 Exam 1 solution Fall 2007

Problem 1 (30 pts)

A continuous integrator with initial condition $x(0)=1$ is shown below.



Fill in the table below where $x(t_n)$ is the exact value of the integrator output at time t_n. Choose $T=0.05$ for each integrator. Round all answers to 4 places after the decimal point.

n	$t_n = nT$	$x_A(n)$ Explicit Euler	$x_A(n)$ Trapezoidal	$x(t_n)$
0	0	1	1	1
1	0.05	1.05	1.0476	1.0476
2	0.1	1.0952	1.0907	1.0906

For a continuous integrator modeled by $\frac{dx}{dt} = f(x,u) = u$,

explicit Euler:

$$x_A(n+1) = x_A(n) + T\cdot f\left[x_A(n), u(n)\right]$$

$$\Rightarrow x_A(n+1) = x_A(n) + T\cdot u(n)$$

$$\Rightarrow x_A(n+1) = x_A(n) + T\cdot e^{-2nT},\quad n=0,1,2,\ldots$$

$n=0$:

$$\begin{aligned} x_A(1) &= x_A(0) + 0.05e^{-2(0)(0.05)} \\ &= 1 + 0.05 \\ &= 1.05 \end{aligned}$$

$n=1$:

$$\begin{aligned} x_A(2) &= x_A(1) + 0.05e^{-2(1)(0.05)} \\ &= 1.05 + 0.0452 \\ &= 1.0952 \end{aligned}$$

trapezoidal: $$x_A(n+1) = x_A(n) + \frac{T}{2} \cdot \left\{ f\left[x_A(n), u(n)\right] + f\left[x_A(n+1), u(n+1)\right] \right\}$$

$$\Rightarrow \quad x_A(n+1) = x_A(n) + \frac{T}{2}\left[u(n) + u(n+1)\right]$$

$$\Rightarrow \quad x_A(n+1) = x_A(n) + \frac{T}{2}\left[e^{-2nT} + e^{-2(n+1)T}\right], \; n = 0,1,2,\ldots$$

$n = 0$: $$x_A(1) = x_A(0) + \frac{0.05}{2}\left[e^{-2(0)(0.05)} + e^{-2(1)(0.05)}\right]$$

$$= 1.0476$$

$n = 1$: $$x_A(2) = x_A(1) + \frac{0.05}{2}\left[e^{-2(1)(0.05)} + e^{-2(2)(0.05)}\right]$$

$$= 1.0907$$

exact soln: $$\frac{dx}{dt} = u(t)$$

$$\int_{x(0)}^{x(t)} dx = \int_0^t u(t)dt$$

$$x(t) - x(0) = \int_0^t e^{-2\lambda} d\lambda$$

$$x(t) = x(0) + \left[\left. \frac{e^{-2\lambda}}{-2} \right|_0^t \right]$$

$$x(t) = 1 + \frac{1 - e^{-2t}}{2}, \; t \geq 0$$

$t = T = 0.05$: $$x(0.05) = 1 + \frac{1 - e^{-2T}}{2} = 1 + \frac{1 - e^{-2(0.05)}}{2} = 1.0476$$

$t = 2T = 0.1$: $$x(0.1) = 1 + \frac{1 - e^{-2(2T)}}{2} = 1 + \frac{1 - e^{-2(0.1)}}{2} = 1.0906$$

Problem 2 (35 pts)

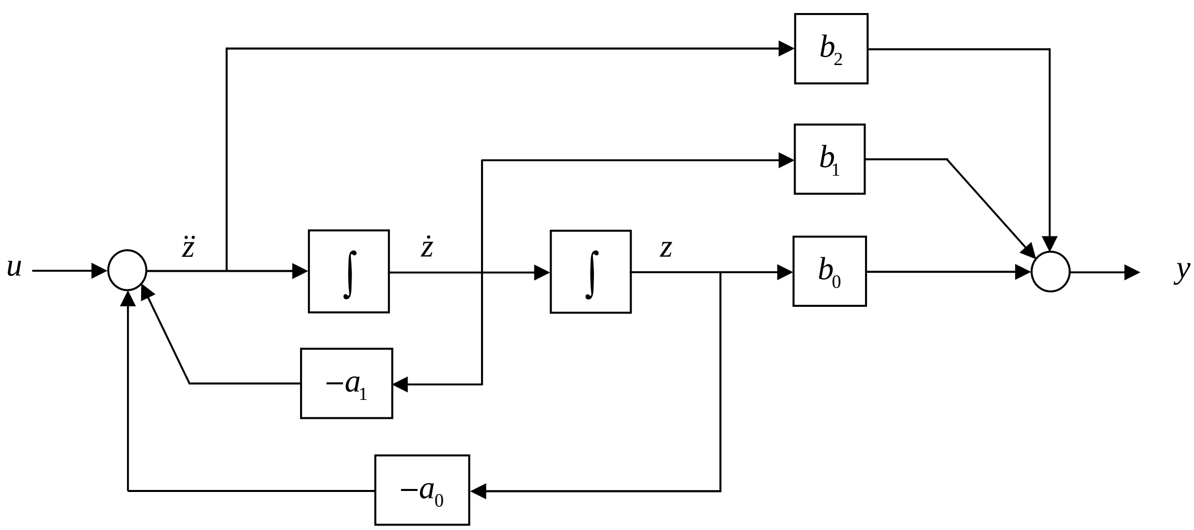
A second order system is modeled by the differential equation

$$\frac{d^2w}{dt^2}+\frac{dw}{dt}+2w=3\frac{d^2u}{dt^2}.$$

The initial conditions $w(0)=\frac{dw}{dt}(0)=0.$

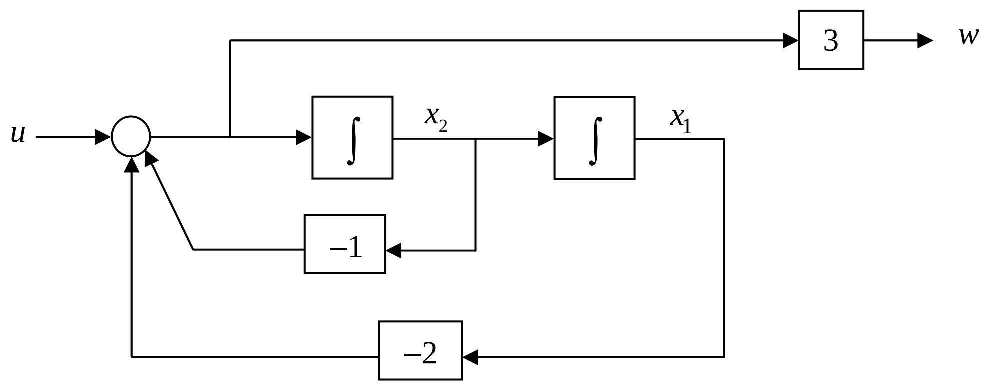
a) Draw a simulation diagram for the system.

$$\frac{d^2}{dt^2}y(t)+a_1\frac{d}{dt}y(t)+a_0y(t) \ =b_0u(t)+b_1\frac{d}{dt}u(t)+b_2\frac{d^2}{dt^2}u(t)$$



$$\frac{d^2w}{dt^2}+\frac{dw}{dt}+2w \ =3\frac{d^2u}{dt^2}$$

$$\Rightarrow \ a_1=1,\ a_0=2,\ b_0=b_1=0,\ b_2=3$$



b) Find matrices A, B, C, D in the state variable model form

$$\underline{\dot{x}} = A\underline{x} + Bu$$
$$y = C\underline{x} + Du$$

The single output is $y = w$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - 2x_1 - x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} y &= w \\ &= 3\dot{x}_2 \\ &= 3(u - 2x_1 - x_2) \\ &= -6x_1 - 3x_2 + 3u \end{aligned}$$

$$y = \begin{bmatrix} -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [3]u$$

$$\Rightarrow \quad C = \begin{bmatrix} -6 & -3 \end{bmatrix}, \qquad D = [3]$$

c) The input $u(t) = t,\ t \ge 0$. Use explicit Euler integration with step size $T = 0.1$ to find $y_A(1), y_A(2)$ and $y_A(3)$. Round answers to 4 places after the decimal point.

Hint: First find $\begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix}, \begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix}, \begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix}$.

$$\dot{x}_1 = f_1[x_1, x_2, u] = x_2$$

$$\dot{x}_2 = f_2[x_1, x_2, u] = -2x_1 - x_2 + u$$

$$x_{1,A}(n+1) = x_{1,A}(n) + Tf_1[x_{1,A}(n), x_{2,A}(n), u(n)]$$

$$\Rightarrow x_{1,A}(n+1) = x_{1,A}(n) + Tx_{2,A}(n)$$

$$x_{2,A}(n+1) = x_{2,A}(n) + Tf_2[x_{1,A}(n), x_{2,A}(n), u(n)]$$

$$\Rightarrow x_{2,A}(n+1) = x_{2,A}(n) + T[-2x_{1,A}(n) - x_{2,A}(n) + u(n)]$$

$$\Rightarrow x_{2,A}(n+1) = -2Tx_{1,A}(n) + (1-T)x_{2,A}(n) + Tu(n)$$

$$\begin{bmatrix} x_{1,A}(n+1) \\ x_{2,A}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ -2T & (1-T) \end{bmatrix} \begin{bmatrix} x_{1,A}(n) \\ x_{2,A}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} u(n)$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(n+1) \\ x_{2,A}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ -2T & (1-T) \end{bmatrix} \begin{bmatrix} x_{1,A}(n) \\ x_{2,A}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} nT$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(n+1) \\ x_{2,A}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -2(0.1) & (1-0.1) \end{bmatrix} \begin{bmatrix} x_{1,A}(n) \\ x_{2,A}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} n(0.1)$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(n+1) \\ x_{2,A}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_{1,A}(n) \\ x_{2,A}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01n \end{bmatrix}$$

$n = 0$:

$$\begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_{1,A}(0) \\ x_{2,A}(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n = 1:$ $$\begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01(1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$n = 2:$ $$\begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01(2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.009 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.029 \end{bmatrix}$$

$$y_A(n) = C\underline{x}_A(n) + Du(n)$$

$$\Rightarrow y_A(n) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} x_{1,A}(n) \\ x_{2,A}(n) \end{bmatrix} + 3u(n)$$

$n = 1:$ $$y_A(1) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} x_{1,A}(1) \\ x_{2,A}(1) \end{bmatrix} + 3nT$$

$$\Rightarrow y_A(1) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3[(1)(0.1)] = 0.3$$

$n = 2:$ $$y_A(2) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} x_{1,A}(2) \\ x_{2,A}(2) \end{bmatrix} + 3nT$$

$$\Rightarrow y_A(2) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + 3[(2)(0.1)] = 0.63$$

$n = 3:$ $$y_A(3) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} x_{1,A}(3) \\ x_{2,A}(3) \end{bmatrix} + 3nT$$

$$\Rightarrow y_A(3) = \begin{bmatrix} -6 & 3 \end{bmatrix} \begin{bmatrix} 0.001 \\ 0.029 \end{bmatrix} + 3[(3)(0.1)] = 0.981$$

Problem 3 (35 pts)

An exponential population growth model

$$\frac{dP}{dt} = -kP, \quad (k > 0)$$

is to be simulated in order to approximate the population $P(t)$ for a period of time. The difference equation for $P_A(n)$ intended to approximate $P(t)$ is

$$P_A(n+1) + \alpha P_A(n) = 0$$

a) Find an expression for α in terms of k and the step size T using

   i) Explicit Euler

   ii) Implicit Euler

b) Evaluate α for each integrator when $k = 0.1$ and $T = 1$ yr. Round answers to 4 places after the decimal point.

c) Fill in the Table below when $P(0) = 5$ million. Round all answers in millions to 4 places after the decimal point.

n	$t_n = nT$	$P_A(n)$, Explicit Euler	$P_A(n)$, Implicit Euler	$P(t_n)$
0	0	5.0000	5.0000	5.0000
1	1			
2	2			

d) The exact solution is given by $P(t) = P(0)e^{-kt}$, $t \ge 0$. Fill in the last column of the table.

a) $$\frac{dP}{dt} = f(P) = -kP$$

i) Explicit Euler:

$$\begin{aligned} P_A(n+1) &= P_A(n) + Tf\left[P_A(n)\right] \\ &= P_A(n) + T\left[-kP_A(n)\right] \\ &= (1-kT)P_A(n) \end{aligned}$$

$$\Rightarrow \quad P_A(n+1) - (1-kT)P_A(n) = 0$$

$$\Rightarrow \quad \alpha = -(1-kT)$$

i) Implicit Euler:

$$\begin{aligned} P_A(n+1) &= P_A(n) + Tf\left[P_A(n+1)\right] \\ &= P_A(n) + T\left[-kP_A(n+1)\right] \end{aligned}$$

$$P_A(n+1) + kTP_A(n+1) = P_A(n)$$

$$\Rightarrow \quad (1+kT)P_A(n+1) - P_A(n) = 0$$

$$\Rightarrow \quad P_A(n+1) - \frac{1}{1+kT}P_A(n) = 0$$

$$\Rightarrow \quad \alpha = -\frac{1}{1+kT}$$

b) Explicit Euler: $\alpha = -(1-kT) = -\left[1-0.1(1)\right] = -0.9$

Implicit Euler: $\alpha = -\frac{1}{1+kT} = -\frac{1}{1+0.1(1)} = -\frac{1}{1.1} = -0.9091$

c) Explicit Euler:

$$P_A(n+1) = 0.9P_A(n), \quad n = 0,1,2,\ldots$$

$$n=0: \quad P_A(1) = 0.9P_A(0) = 0.9(5) = 4.5$$

$$n=1: \quad P_A(2) = 0.9P_A(1) = 0.9(4.5) = 4.05$$

Implicit Euler:

$$P_A(n+1) = 0.9091P_A(n), \quad n = 0,1,2,\ldots$$

$$n=0: \quad P_A(1) = 0.9091P_A(0) = 0.9091(5) = 4.5455$$

$$n=1: \quad P_A(2) = 0.9091P_A(1) = 0.9091(4.5455) = 4.1322$$

n	$t_n = nT$	$P_A(n)$, Explicit Euler	$P_A(n)$, Implicit Euler	$P(t_n)$
0	0	5.0000	5.0000	5.0000
1	1	4.5000	4.5455	4.5242
2	2	4.0500	4.1322	4.0937

d)

$$P(t) = P(0)e^{-kt}, \quad t \geq 0$$

$$\Rightarrow P(t_n) = P(0)e^{-kt_n}$$

$$\Rightarrow P(t_n) = P(0)e^{-knT}, \quad n = 0,1,2,\ldots$$

$n = 0$:

$$P(t_0) = 5e^{-0.1(0 \cdot T)} = 5$$

$n = 1$:

$$P(t_1) = 5e^{-0.1(1 \cdot T)} = 5e^{-0.1(1 \cdot 1)} = 4.5242$$

$n = 2$:

$$P(t_2) = 5e^{-0.1(2 \cdot T)} = 5e^{-0.1(2 \cdot 1)} = 4.0937$$