EEL 4890 Exam 1 solution Name________________

Fall 2011

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Problem 1 (50 pts)

An unforced linear first order system is described by $\frac{dx}{dt}+\lambda x=0.$ The initial state is $x(0)=1.$

a) Find the state derivative function $f(x,u)$. (5 pts)

$$f(x,u)=\frac{dx}{dt}=-\lambda x$$

b) The difference equation for approximating the state using explicit Euler integration reduces to $x_A(n+1)=\beta x_A(n),\ n=0,1,2,...$. Find the expression for β in terms of λ and T. (5 pts)

$$\begin{aligned}
& x_A(n+1) = x_A(n)+Tf[x_A(n),u(n)] \\
\Rightarrow\ & x_A(n+1) = x_A(n)+T[-\lambda x_A(n)] \\
\Rightarrow\ & x_A(n+1) = x_A(n)-\lambda T x_A(n) \\
\Rightarrow\ & x_A(n+1) = (1-\lambda T)x_A(n) \\
\Rightarrow\ & \beta = 1-\lambda T
\end{aligned}$$

c) The difference equation for approximating the state using trapezoidal integration reduces to $x_A(n+1)=\beta x_A(n),\ n=0,1,2,...$. Find the expression for β in terms of λ and T. (10 pts)

$$\begin{aligned}
& x_A(n+1) = x_A(n)+\frac{T}{2}\left\{f[x_A(n),u(n)]+f[x_A(n+1),u(n+1)]\right\} \\
\Rightarrow\ & x_A(n+1) = x_A(n)+\frac{T}{2}\left\{[-\lambda x_A(n)]-\lambda x_A(n+1)\right\} \\
\Rightarrow\ & x_A(n+1) = x_A(n)-\frac{\lambda T}{2}\left\{x_A(n)+x_A(n+1)\right\} \\
\Rightarrow\ & x_A(n+1) = \left(1-\frac{\lambda T}{2}\right)x_A(n)-\frac{\lambda T}{2}x_A(n+1) \\
\Rightarrow\ & \left(1+\frac{\lambda T}{2}\right)x_A(n+1) = \left(1-\frac{\lambda T}{2}\right)x_A(n)
\end{aligned}$$

$$\Rightarrow \quad x_A(n+1) = \frac{\left(1-\frac{\lambda T}{2}\right)}{\left(1+\frac{\lambda T}{2}\right)} x_A(n)$$

$$\Rightarrow \quad \beta = \frac{\left(1-\frac{\lambda T}{2}\right)}{\left(1+\frac{\lambda T}{2}\right)} = \frac{2-\lambda T}{2+\lambda T}$$

d) The exact solution is given by $x(t) = x(0)e^{-\lambda t}$. Fill in the table below for $\lambda = 2$ and $T = 0.05$ for each integrator. Round all answers to 4 places after the decimal point. (15 pts)

explicit Euler: $\quad x_A(n+1) = (1-\lambda T)x_A(n) = [1-2(0.05)]x_A(n) = 0.9x_A(n)$

$n=0$: $\quad x_A(1) = 0.9x_A(0) = 0.9(1) = 0.9$

$n=1$: $\quad x_A(2) = 0.9x_A(1) = 0.9(0.9) = (0.9)^2 = 0.81$

$n=2$: $\quad x_A(3) = 0.9x_A(2) = 0.9(0.81) = (0.9)^3 = 0.7290$

trapezoidal: $\quad x_A(n+1) = \frac{\left(1-\frac{\lambda T}{2}\right)}{\left(1+\frac{\lambda T}{2}\right)} x_A(n) = \frac{\left[1-\frac{2(0.05)}{2}\right]}{\left[1+\frac{2(0.05)}{2}\right]} x_A(n) = \frac{0.95}{1.05} x_A(n) = \frac{19}{21} x_A(n)$

$n=0$: $\quad x_A(1) = \frac{19}{21} x_A(0) = \frac{19}{21}(1) = \frac{19}{21} = 0.9048$

$n=1$: $\quad x_A(2) = \frac{19}{21} x_A(1) = \frac{19}{21}\left(\frac{19}{21}\right) = \left(\frac{19}{21}\right)^2 = 0.8186$

$n=2$: $\quad x_A(3) = \frac{19}{21} x_A(2) = \frac{19}{21}\left(\frac{19}{21}\right)^2 = \left(\frac{19}{21}\right)^3 = 0.7406$

$$x(t) = x(0)e^{-\lambda t} = e^{-2t}$$

$n=0,\ t_0=0$: $\quad x(t_0) = x(0) = e^{-2(0)} = 1$

$n=1,\ t_1=T=0.05$: $\quad x(t_1) = x(T) = x(0.05) = e^{-2(0.05)} = e^{-0.1} = 0.9048$

$n=2,\ t_2=2T=0.1$: $\quad x(t_2) = x(2T) = x(0.1) = e^{-2(0.1)} = e^{-0.2} = 0.8187$

$n=3,\ t_1=3T=0.15$: $\quad x(t_3) = x(3T) = x(0.15) = e^{-2(0.15)} = e^{-0.3} = 0.7408$

n	$t_n = nT$	$x_A(n)$ Explicit Euler	$x_A(n)$ Trapezoidal	$x(t_n)$
0	0	1	1	1
1	0.05	0.9	0.9048	0.9048
2	0.1	0.81	0.8186	0.8187
3	0.15	0.729	0.7406	0.7408
50	2.5	0.0052	0.0067	0.0067

e) Based on the answers to Part d), find the general solution for $x_A(n),\ n = 0,1,2,\ldots$ for the explicit Euler and trapezoidal integrators. (10 pts)

explicit Euler: $x_A(n) = (0.9)^n,\ n = 0,1,2,\ldots$

trapezoidal: $x_A(n) = \left(\frac{19}{21}\right)^n,\ n = 0,1,2,\ldots$

f) Find $x_A(50)$ for the explicit Euler and trapezoidal integrators and the exact solution $x(50T)$. Enter the results in the last row of the table. (5 pts)

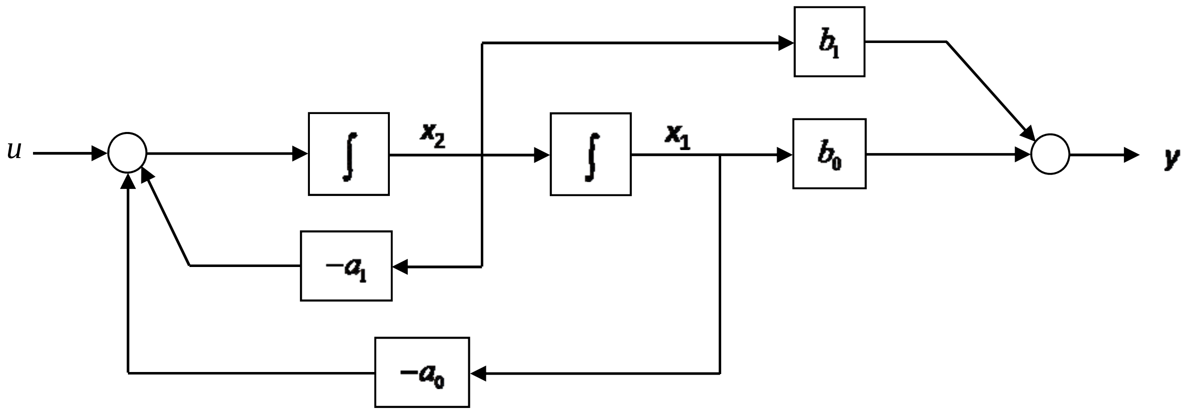
explicit Euler: $x_A(50) = (0.9)^{50} = 0.0052$

trapezoidal: $x_A(50) = \left(\frac{19}{21}\right)^{50} = 0.0067$

exact soln: $x(50T) = x(2.5) = e^{-2(2.5)} = e^{-5} = 0.0067$

Problem 2 (25 pts)

A simulation diagram for a second order system is shown below.



a) Find the matrices A, B, C, D in the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu, \quad y = C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - a_0 x_1 - a_1 x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = b_0 x_1 + b_1 x_2$$

$$= \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

$$\Rightarrow C = \begin{bmatrix} b_0 & b_1 \end{bmatrix}, \quad D = [0]$$

Problem 3 (25 pts)

A population growth model

$$\frac{dP}{dt} = \frac{\alpha}{P}, \quad \alpha = 0.2 \times 10^6$$

is to be simulated using explicit Euler integration in order to approximate the population $P = P(t)$ for a period of time. The initial population, $P_0 = 1{,}000$ and $T = 0.5$ yr.

a) Find the difference equation for $P_A(n)$ intended to approximate the true population $P(nT),\ n = 0,1,2,\ldots$. (10 pts)

$$\frac{dP}{dt} = f(P) = \frac{\alpha}{P}$$

explicit Euler: $\quad P_A(n+1) = P_A(n) + Tf\left[P_A(n)\right]$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + T\left[\frac{\alpha}{P_A(n)}\right]$$

$$\Rightarrow \quad P_A(n+1) = P_A(n) + \frac{\alpha T}{P_A(n)}$$

b) Find the approximate population (to the nearest integer) at the end of 1 yr. (10 pts)

$$P_A(n+1) = P_A(n) + T\left[\frac{\alpha}{P_A(n)}\right]$$

$n = 0$: $\quad P_A(1) = P_A(0) + T\left[\frac{\alpha}{P_A(0)}\right] = P(0) + \frac{\alpha T}{P(0)} = 1000 + \frac{(0.2\times 10^6)(0.5)}{1000} = 1000 + 100 = 1100$

$n = 1$: $\quad P_A(2) = P_A(1) + T\left[\frac{\alpha}{P_A(1)}\right] = P_A(1) + \frac{\alpha T}{P_A(1)} = 1100 + \frac{(0.2\times 10^6)(0.5)}{1100} = 1100 + 90.9 = 1190.9$

$$P_A(2) \approx 1191$$

c) The exact solution for the population is $P(t) = \left[(P_0)^2 + 2\alpha t\right]^{1/2},\ t \ge 0$. Find the exact population (to the nearest integer) at the end of 1 yr. (5 pts)

$$P(t) = \left[(P_0)^2 + 2\alpha t\right]^{1/2}, \quad t \ge 0$$

$$P(1) = \left[(1000)^2 + 2(0.2\times 10^6)(1)\right]^{1/2} = 1183.2$$

$$P(1) \approx 1183$$