The population of a country was measured every ten years and the results are shown below.

| $t$ | $P_{obs}$, Population (in thousands) |
|---|---|
| 0 | 9.6 |
| 10 | 18.3 |
| 20 | 29.0 |
| 30 | 47.2 |
| 40 | 71.1 |
| 50 | 119.1 |
| 60 | 174.6 |
| 70 | 257.3 |
| 80 | 350.7 |
| 90 | 441.0 |
| 100 | 513.3 |
| 110 | 559.7 |
| 120 | 594.8 |
| 130 | 629.4 |
| 140 | 640.8 |
| 150 | 651.1 |
| 160 | 655.9 |
| 170 | 659.6 |
| 180 | 661.8 |
| 190 | 663.7 |
| 200 | 665.0 |

A logistic growth model is to be fit to the data. The carrying capacity "$M$" is estimated to be 665 (in thousands).

a) Estimate the parameter "$b$" in the logistic growth model

$$\frac{dP}{dt} = g(P)P = b(M-P)P$$

where $P = P(t)$ is the population at time "$t$" and $g(P)$ is the population-dependent growth rate.

Hint: Use the analytical solution of the differential equation model,

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-bMt}}$$

and solve for the parameter "$b$" in terms of $M, P_0$, and $t$. Generate a set of nineteen values of "$b$" by substituting $t = 10, 20, 30, \ldots, 190$ into the expression for "$b$". Use the average of the nineteen "$b$" values as your estimate of "$b$".

b) Using the estimated values of "$M$" and "$b$", simulate the logistic population growth for 200 years using forward Euler integration with step sizes of $T = 10, 5, 1, 0.1$ years. Use the Matlab "subplot" to produce 4 plots (2 by 2) in the same Figure Window. In addition to labeled axes and a title, each plot will include:

i) the observed population data (as black dots)

ii) the analytical solution (as a solid red curve)

iii) the simulated solution (as a dotted blue line thru the simulated points)

iv) the step size $T$

v) A legend with the text "obs", "sim", and "anal" to distinguish the observed data points, analytical solution and simulated solution

Comment briefly on the results.

c) Perform a sensitivity analysis of the analytical logistic population growth with respect to changes in the estimated parameter "$b$". In other words, do a multi-run with "$b$" varying by 50% in either direction.

Plot the analytical population vs. time corresponding to five equally spaced values of "$b$" ranging from $0.5 \times$ "estimate of $b$" up to $1.5 \times$ "estimate of $b$". In addition to labeled axes and a title, the graph should include the minimum and maximum values of "$b$" displayed near the corresponding curves.

Comment briefly on the significance of the parameter "$b$".

d) Using the original estimate of "$b$", plot the growth rate $g(p)$ vs $P$ over the range of values $0 \le P \le 2M$. In addition to labeled axes and a title, the graph should include

i) the points on the growth rate $g(P)$ vs $P$ corresponding to the times $t = 0, 40, 80, 120, 160$. Show the points as red dots.

ii) the point on the growth rate $g(P)$ vs $P$ corresponding to ZPG (zero population growth). Show the point as a black dot.

e) Prepare a Simulink diagram for simulating the logistic growth model. Choose the Euler integrator (ode1), step size $T = 0.1\ \mathrm{yr}$ and simulate the population growth for 200 years. Include Simulink blocks for computing the analytical solution $P(t)$. In addition, include a "From Workspace" block from the "Sources" sublibrary and set the "Data" parameter equal to the array "t_Pobs".

The Matlab script file must include the following statements:

```
t=0:10:200;
Pobs=[9.6 18.3 29.0 47.2 71.1 119.1 174.6 257.3 350.7 441.0 513.3 559.7 594.8 629.4
640.8 651.1 655.9 659.6 661.8 663.7 665.0];
t_Pobs=[t; Pobs]';
sim('Logistic_growth')
```

Configure a scope for 3 inputs and feed it (from top to bottom) Pobs, Psim and Panal.

Run the script file, open the scope and paste the scope with the 3 signals into your report.

Comment on the results.