The strength of a homogeneous tank force is determined by the number of tanks in operation at time *t*, denoted *x*(*t*). Opposing forces consist of foot soldiers armed with antitank weapons. The strength of anti-tank weapon forces is measured by the number of weapons operational at time *t*, denoted *y*(*t*). The basic combat model is:

$$\frac{d}{dt}x(t) = -ay(t), \quad (a > 0)$$

$$\frac{d}{dt}y(t) = -bx(t), \quad (b > 0)$$

where *a* = Antitank weapon attrition coefficient
b = Tank attrition coefficient

Note, "*a*" and "*b*" are casualty rates per unit of opposing forces, i.e.

$$a = \frac{\left|\frac{dx}{dt}\right|}{y}, \qquad b = \frac{\left|\frac{dy}{dt}\right|}{x}$$

"*a*" is a measure of how effectively a single antitank weapon can destroy tanks and "*b*" measures the same with respect to a single tank's ability to destroy antitank weapons.

Given numerical values for "*a*" and "*b*" as well as initial force strengths $x_0 = x(0)$ and $y_0 = y(0)$, the basic model is called a "Lanchester-type combat model".

a) Prepare a SIMULINK block diagram for simulating a battle between tanks and antitank forces. Use the ode4 integrator with step size $T = 0.01$ days.

   Hint: Set the simulation final time to 10 days and use a “STOP” block to halt the simulation when either side is completely depleted.

b) The solution trajectories in the *x*-*y* plane satisfy

$$ay^2 - bx^2 = c \quad \text{where} \quad c = ay_0^2 - bx_0^2$$

   The final outcome of combat between forces is

$$y = 0, \qquad x = \left(-\frac{c}{b}\right)^{1/2} \qquad \text{for } c < 0$$

$$y = 0, \qquad x = 0 \qquad \text{for } c = 0$$

$$y = \left(\frac{c}{a}\right)^{1/2}, \qquad x = 0 \qquad \text{for } c > 0$$

Simulate combat between the forces for the following case:

$x_0 = 1000$ tanks, $y_0 =$ antitank 2000 weapons,

$a = 0.1$ tanks/day per antitank weapon, $b = 0.2$ antitank weapons/day per tank

Compare the simulated and analytical solution trajectory $y(t)$ vs $x(t)$.
Plot both on the same graph with the analytical solution as a solid blue curve and every 20th point on the simulated response as a red data point.

c) The analytical solutions to the Lanchester combat model are:

$$x(t) = x_0 \cosh\left(\delta t\right) - y_0 \frac{\delta}{b}\sinh(\delta t)$$

$$y(t) = y_0 \cosh\left(\delta t\right) - x_0 \frac{\delta}{a}\sinh(\delta t)$$

where $\sinh(\delta t) = \tfrac{1}{2}\left(e^{\delta t} - e^{-\delta t}\right)$, $\cosh(\delta t) = \tfrac{1}{2}\left(e^{\delta t} + e^{-\delta t}\right)$ and $\delta = (ab)^{1/2}$

i) Graph the simulated and analytical time histories of $x(t)$ and $y(t)$ on the same graph for the case in Part b). Plot the analytical solution for $x(t)$ as a solid black curve and the analytical solution for $y(t)$ as a solid blue curve. Plot every 20th point of the simulated responses for $x(t)$ and $y(t)$ as a red data point.

ii) Compare the simulated and analytical final outcomes.

d) The time it takes for the antitank forces to defeat the tank force is given by

$$T = \frac{1}{2\delta}\ln\left(\frac{\delta y_0 + bx_0}{\delta y_0 - bx_0}\right)$$

Verify the result using simulation for the case in Part b).

e) When a weapon system employs "area fire" and opposing forces defend a constant area, the attrition rate coefficients depend on the number of targets, i.e. opposing force strengths and

$$a = a(x) = gx \qquad \Rightarrow \qquad \frac{d}{dt}x(t) = -gxy \qquad (g > 0)$$

$$b = b(y) = hy \qquad \Rightarrow \qquad \frac{d}{dt}y(t) = -hxy \qquad (h > 0)$$

This model is indicative of combat between two guerrilla forces. The solution trajectories satisfy:

$$gy - hx = k \quad \text{where} \quad k = gy_0 - hx_0$$

The final outcome of combat between forces is

$$x = 0, \qquad y = \frac{k}{g} \qquad \text{if } k > 0$$

$$x = 0, \qquad y = 0 \qquad \text{if } k = 0$$

$$x = -\frac{k}{h}, \qquad y = 0 \qquad \text{if } k < 0$$

Prepare a Simulink diagram for simulating a battle for 10 days between tanks and antitank forces for the case

$x_0 = 6000$ tanks, $y_0 = 1500$ antitank weapons

$g = 0.00075$ tanks/day per antitank weapon per tank,

$h = 0.00025$ antitank weapons/day per tank per antitank weapon

i) Obtain a simulated solution trajectory $y(t)$ vs $x(t)$ and compare it to the analytical solution trajectory. Use the subplot feature in Matlab and plot the simulated solution trajectory in the top half and the analytical solution trajectory in the bottom half.

ii) Plot the simulated time history $x(t)$ vs t as a solid blue curve and the simulated time history $y(t)$ vs t as a solid red curve on the same graph.

iii) Compare the final simulated outcomes to the analytical results.

Comment on the results.

f) The analytical solution for $x(t)$ is

$$x(t) = \begin{cases} x_0 \left[\dfrac{hx_0 - gy_0}{hx_0 - gy_0 e^{\left[-(hx_0 - gy_0)t\right]}} \right] & \text{for } hx_0 \neq gy_0 \\[2ex] \dfrac{x_0}{1 + hx_0 t} & \text{for } hx_0 = gy_0 \end{cases}$$

Compare the simulation and analytical solutions for $x(t)$ for the case

$x_0 = 2000$ tanks, $y_0 = 3000$ antitank weapons

$g = 0.0003$ tanks/day per antitank weapon per tank,

$h = 0.0002$ antitank weapons/day per tank per antitank weapon

Plot both on the same graph with the analytical solution as a solid blue curve and every 20th point on the simulated response as a red data point.