Function Practice Problems from old AHSME/AMC/AIME

1) (1993 AHSME #12) If $f(2x) = \frac{2}{2+x}$, for all x > 0, what is 2f(x)?

Solution

Plug in x/2: $f\left(2\left(\frac{x}{2}\right)\right) = \frac{2}{2+\frac{x}{2}} = \frac{2}{\frac{2(2)+x}{2}} = \frac{4}{x+4}$. Thus, 2f(x) =$\boldsymbol{\frac{8}{x+4}}$.

2) (1995 AHSME #14) If f(x) = $ax^4 - bx^2 + x + 5$ and f(-3) = 2, then what is f(3)?

Solution

$f(-3) = a(-3)^4 - b(-3)^2 + -3 + 5 = 2$, so 81a - 9b + 2 = 2 → 81a-9b = 0
$f(3) = a(3)^4 - b(-3)^2 + 3 + 5 = 81a - 9b + 8 = 0 + 8 = 8$
f(3) = 8.

3) (1988 AHSME #15) If a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, then what is b?

Solution

For some function f(x), we have: $ax^3 + bx^2 + 1 = (x^2 - x - 1)f(x)$

Notice that for the first term to match, f(x) = ax + c, for constants a and c:

$ax^3 + bx^2 + 1 = (x^2 - x - 1)(ax+c)$
$ax^3 + bx^2 + 1 = ax^3 - ax^2 - ax + cx^2 - cx - c$
$ax^3 + bx^2 + 1 = ax^3 + (c - a)x^2 - (a+c)x - c$

Equate coefficients.
When equating the constant coefficients we have +1 = -c, so c = -1.
When equating coefficients for x we have 0 = -(a+c), so c = -a, a = 1
When equating coefficients for x^2 we have b = c – a = -1 – 1 = -2.
Thus, **b = -2.**

4) (1984 AHSME #16) The function f(x) satisfies f(2 + x) = f(2 – x) for all real numbers x. If the equation f(x) = 0 has four distinct real roots, what is the sum of those roots?

Solution

Let two of those distinct roots be a and b, where both a > 2 and b > 2. If a is a root greater than 2, then we can write f(a) = f(2 + x), so a = 2 + x and x = a – 2. 0 = f(2 + x) = f(2 – x) = f(2 – (a – 2)) = f(4 – a).

Thus, if a is one root, 4 – a is another distinct root.
Similarly, if b is one root, 4 – b is another distinct root.
Thus, the sum of these four roots must be a + (4 – a) + b + (4 – b) = **8.**

5) (1983 AHSME #18) Let f be a polynomial function such that for all real x, $f(x^2+1) = x^4 + 5x^2 + 3$. For all real x, what is $f(x^2-1)$?

Solution

There exist constants a and b such that

$$f(x^2+1) = (x^2+1)^2 + a(x^2+1) + b = x^4 + 5x^2 + 3$$

$$x^4 + 2x^2 + 1 + ax^2 + a + b = x^4 + 5x^2 + 3$$

$$x^4 + (2+a)x^2 + (a+b+1) = x^4 + 5x^2 + 3$$

Equate the coefficient for x^2, so we have 2 + a = 5, so a = 3.
Equate constant coefficients, so we have a + b + 1 = 3. Since a=3, b=-1.
It follows that f(x) = x^2 + 3x -1.

Thus, $f(x^2-1) = (x^2-1)^2 + 3(x^2-1) - 1$
$= x^4 - 2x^2 + 1 + 3x^2 - 3 - 1 = \boldsymbol{x^4 + x^2 - 3}$.

6) (1999 AHSME #17) Let P(x) be a polynomial such that when P(x) is divided by x – 19, the remainder is 99, and when P(x) is divided by x – 99, the remainder is 19. What is the remainder when P(x) is divided by (x – 19)(x – 99)?

Solution

There exists polynomials S(x) and T(x), with T(x) being degree one such that:

P(x) = S(x)(x – 19)(x – 99) + T(x)

We know that P(19) = 99 and P(99) = 19.

P(19) = S(19)(0) + T(19) = 99
P(99) = S(99)(0) + T(99) = 19

We also know that T(x) = ax + b, for constants a and b, since T(x) has to be a function with a degree lower than (x – 19)(x – 99).

T(19) = 19a + b = 99
T(99) = 99a + b = 19

Subtract top equation from bottom to yield 80a = - 80, a = -1, b = 118
Thus, the remainder is **–x + 118.**

7) (1991 AHSME #21) If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$, for all $x \neq 0,1,1$ and $0 < \theta < \frac{\pi}{2}$, then what is $f(sec^2 x)$?

Solution

$$f\left(1 + \frac{1}{x-1}\right) = \frac{1}{x}$$

I need to figure out what to plug into the expression inside of the f to make it equal to x.

So we want to solve for y in terms of x in the equation below.

$$x = 1 + \frac{1}{y-1}$$
$$x - 1 = \frac{1}{y-1}$$
$$y - 1 = \frac{1}{x-1}$$
$$y = 1 + \frac{1}{x-1}$$

So,

$$f\left(1 + \frac{1}{1 + \frac{1}{x-1} - 1}\right) = \frac{1}{1 + \frac{1}{x-1}}$$

$$f(x) = \frac{1}{1 + \frac{1}{x-1}} = \frac{1}{\frac{x-1+1}{x-1}} = \frac{x-1}{x}$$

Now we can solve for $f(sec^2x)$:

$$f(sec^2x) = \frac{sec^2x - 1}{sec^2x} = \frac{tan^2x}{sec^2x} = (\frac{tanx}{secx})^2 = (\frac{\frac{sinx}{cosx}}{\frac{1}{cosx}})^2$$

$$= (\frac{sinxcosx}{cosx})^2 = sin^2x$$

8) (1986 AHSME #24) Let $p(x) = x^2 + bx + c$, where b and c are integers. If p(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, what is p(1)?

Solution

If p(x) is a factor of both polynomials it would also be a factor of 3f(x) – g(x).

$3f(x) = 3x^4 + 18x^2 \qquad + 75$
$-g(x) = 3x^4 + 4x^2 + 28x + 5$

$3f(x)-g(x) = 14x^2 – 28x + 70 = 14(x^2 – 2x + 5)$

p(x) must divide into $14(x^2 – 2x + 5)$ evenly, which means that $p(x) = x^2 – 2x + 5$. It follows that p(1) = 1 – 2 + 5 **= 4.**

9) (2003 AMC A #20) If $f(x) = ax^3 + bx^2 + cx + d$ and f(-1) = 0, f(0) = 2 and f(1) = 0, what is b?

Solution

f(-1) = -a + b – c + d = 0
f(0) = d = 2
f(1) = a + b + c + d = 0

f(-1) = -a + b – c + 2 = 0
f(1) = a + b + c + 2 = 0

Now add the two equations:

f(-1) + f(1) = 2b + 4 = 0. It follows that b = **-2.**

10) (1993 AIME #5) Let $P_0(x) = x^3 + 313x^2 – 77x – 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

Solution

$P_n(x) = P_{n-1}(x - n) = P_{n-2}(x – n – (n – 1)) = \ldots P_0(x – (n + (n-1)+\ldots 1))$

$P_n(x) = P_0(x - \frac{n(n+1)}{2})$

So, $P_{20}(x) = P_0\left(x - \frac{20(21)}{2}\right) = P_0(x - 210)$

$$P_0(x - 210) = (x - 210)^3 + 313(x - 210)^2 - 77(x - 210) - 8$$

Now, we must find the coefficient of x in what is above:

$3(210)^2 – 2(313)(210) – 77$

(630) (210) – (626)(210) – 77 = (210)(630 – 626) – 77 = 4(210) – 77 = 840 – 77 = **763.**

11) (1986 AIME #11) The polynomial $1 – x + x^2 – x^3 + \ldots + x^{16} – x^{17}$ may be written in the form $a_0 + a_1y + a_2y^2 + \cdots + a_{17}y^{17}$, where y = x+1 and all the ai's are constants. Find the value of a_2.

Substitute y = x + 1, means that x = y – 1:

$f((y-1)) = 1 - (y-1) + (y-1)^2 - (y-1)^3 + \ldots + (y-1)^{16} - (y-1)^{17}$

Notice that –(y-1) = 1 – y, so we have
$f(1-y) = 1 + (1 – y) + (1 – y)^2 + (1 – y)^3 + \ldots + (1 – y)^{17}$

Coefficient of $y^2 = \sum_{k=2}^{17} \binom{k}{2}$

Using the hockey stick identity by repeatedly applying Pascal's Triangle we find that this sum equals $\binom{18}{3} = \frac{18\times17\times16}{6} = 3 \times 17 \times 16 = \mathbf{816}$.

Exercise left to the reader: Use induction on n to prove the following for all integers greater than or equal to a. (Note: a is a fixed integer.)

$$\sum_{k=a}^{n} \binom{k}{a} = \binom{n+1}{a+1}$$

Functions – Some Key Points

1) You can plug in anything you want for x in f(x) to obtain the information that you need.

2) Exploit the fact that $x^{2n} = (-x)^{2n}$, when evaluating f(x) and f(-x). So, of you are give f(x) and most of the terms have even powers, it helps you find f(-x), since those even powered terms stay the same.

3) When you have two different expressions for the same polynomial, equate coefficients.

4) When there is a vertical line of symmetry, roots come in pairs (except if the line of symmetry contains a root. Thus, the sum of the roots is whatever that line of symmetry is (x = c for some constant c) times the number of roots.

5) When given that the remainder of P(x) divided by x – r is q, this means that P(r) = q.

6) Just like with integers, if a polynomial divides into two other polynomials, it also divides into any linear combination of those two polynomials.